

MATHEMATICS teacher

MARCH 2014



homework

the filling between question and answer

PROBLEM POSING
a rationale for e

TRAPEZOIDS
decomposed and transformed

~~CURRENT YR/VOL~~

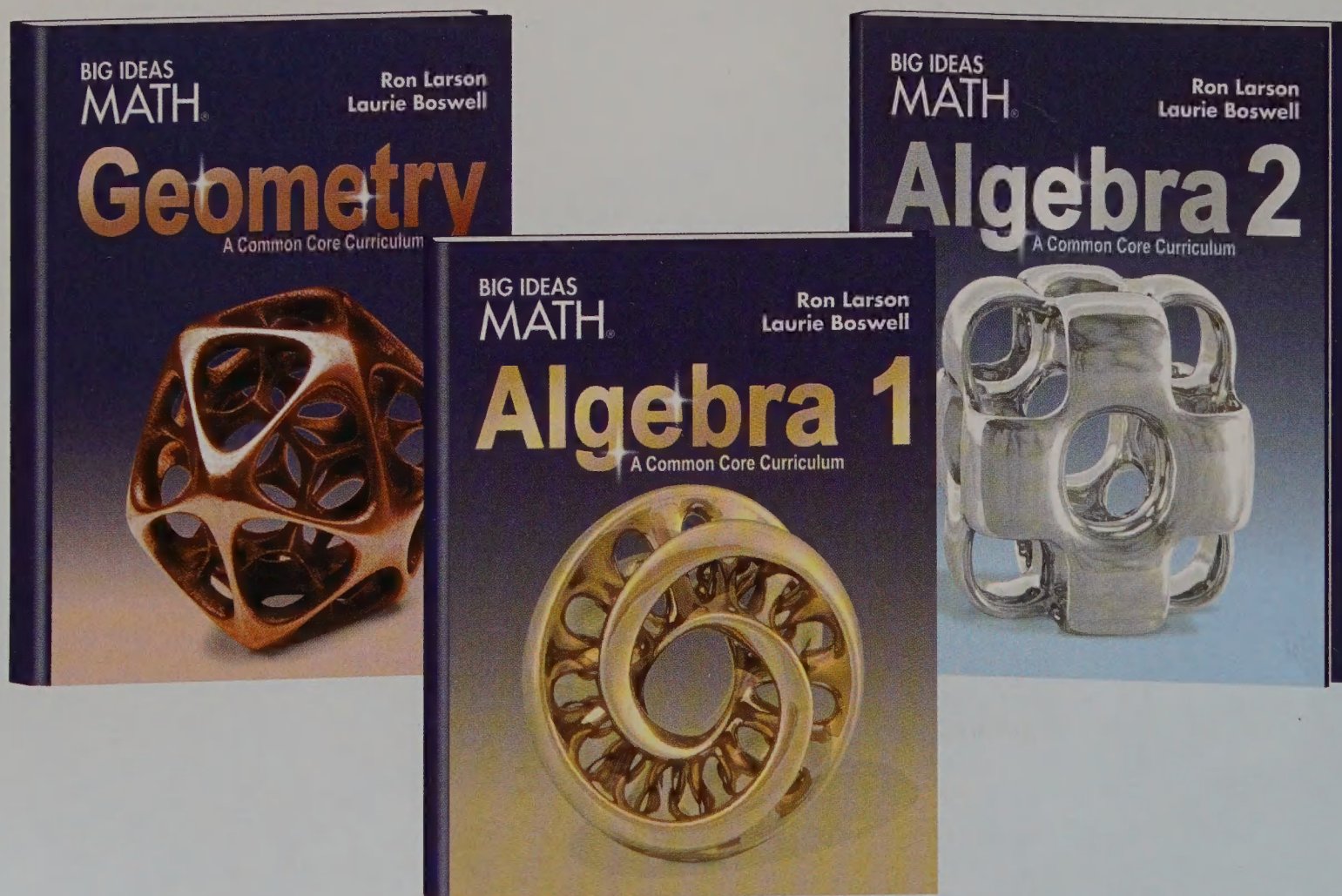
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on the cover

“Hooray for mistakes!” Finding them, fixing them, and understanding them are all valuable parts of the learning process. In “Algebra Homework: A Sandwich!” (pp. 528–33), D. Bruce Jackson reflects on a teacher’s strategy to make homework productive for all students. Ms. Warburton’s system empowers her students with confidence and continuous improvement. That’s something to cheer about.

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- Activities for Students (p. 547): Customizable activity sheets plus solutions to the activities



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Coming in April 2014:

- "Not So Complex: Iteration in the Complex Plane," by Robin S. O'Dell
- "Graphing Inequalities, Connecting Meaning," by J. Matt Switzer
- "Investigating Home Primes and Their Families," by Marlena Herman and Jay Schiffman

MORE ON APPORTIONMENT

Regarding “What If? How Apportionment Methods Choose Our Presidents,” by Michael J. Caulfield (*MT* Oct. 2012, vol. 106, no. 3, pp. 178–83), readers might be interested in one aspect of the debate about the first U.S. congressional apportionment bill of 1792, which became the subject of George Washington’s first (and successful) veto. The issue was not which apportionment method to use; it was how to apply the Constitution’s ratio test.

Washington’s secretary of the treasury, Alexander Hamilton, who had proposed the bill, felt that it met the constitutional requirement because its apportionment numbers did not exceed “one [representative] for every thirty Thousand” eligible residents in the *aggregate*; that is, the ratio of the total of all representatives to the total number of U.S. eligible residents met the requirement. Secretary of State Thomas Jefferson opposed the bill because the apportionment numbers exceeded the required ratio for some states *individually*. (See, e.g., such websites as <http://gwpapers.virginia.edu/documents/presidential/veto.html>.)

In his veto message of April 5, Washington responded, “The Constitution has also provided that the number of Representatives shall not exceed one for every thirty thousand: which restriction is, by the context and by fair and obvious construction, to be applied to the separate and respective numbers of the States: and the bill has allotted to eight

of the States more than one for thirty thousand.”

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Hartford, CT, July 31, 2013

A SMOOTH RIDE FOR SQUARE-WHEELED TRICYCLES

In “A Road for Every Wheel,” about a square-wheeled tricycle (*Mathematical Lens*, *MT* Nov. 2012, vol. 106, no. 4, pp. 254–59), I interpreted the *New York Times* article to mean that the underlying surface on which the tricycle rode rose and fell to keep the axles at the same height as the tricycle was ridden around the circular track. A check of the article appears that my interpretation was correct:

Two years ago, [Glen Whitney] and his team built a carnival-like traveling exhibit called the Math Midway, a proof-of-concept for the coming museum [the Museum of Mathematics in New York City]. It includes a tricycle with square wheels of different sizes that visitors can ride smoothly around a circular path ridged like a flower’s petals. An accompanying sign explains why: The undulating circular surface rises and falls exactly to offset the odd shape of the wheels, so that the tricycle’s axles—and the rider—remain at the same height as they move. (Kenneth Chang, “One Math Museum, Many Variables,” *New York Times*, June 28, 2011, *Science Times*, p. 1)

I have since realized that this passage can be interpreted in another way—by assuming that the underlying surface on which the tricycle rides is not a cycloid but a catenary (John D. Barrow, *100 Essential Things You Didn’t Know You Didn’t Know: Math Explains Your World*, W.W. Norton, 2008, pp. 174–75). If that is the case, the axle height remains constant as the square wheels climb over the catenary surface.

For purposes of illustration, assume that the underlying surface has the equation $y = -(e^x + e^{-x})/2 = -\cosh(x)$. The curve is symmetric with respect to the y -axis, has y -intercept -1 , and is “upside down” so that the square-wheeled tricycle can be ridden on it. If this is the equation of the underlying surface, then $dy/dx = -(e^x - e^{-x})/2 = -\sinh(x)$ is the slope of the tangent line. At the notches where sections of catenaries meet, a right angle is formed, allowing the corner of the square wheel to fit into it. Thus, at that cusp, the two tangent lines have slopes of ± 1 . Solving this for x , we get $x \approx \pm 0.88$.

The arc length of the curve is given by

$$S = \int_{-a}^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_{-a}^a \sqrt{1 + \left(-\frac{1}{2}(e^x - e^{-x})\right)^2} dx,$$

which simplifies to $S = e^a - e^{-a}$ (or $\sinh(a) - \sinh(-a)$). For $a \approx \pm 0.88$, the arc length is 2—the length of the side of the square wheel.

At $x \approx \pm 0.88$, the y -coordinate of the curve is $-\sqrt{2}$. But the section of the diagonal running from the corner in the notch to the axle is $\sqrt{2}$; thus, the axle is at $y = 0$. At $x = 0$, the y -coordinate of the catenary is -1 . However, with a horizontal tangent at that point, the distance to the axle is 1, again giving the axle height as $y = 0$.

For any point $B = (b, -\cosh(b))$ on the curve and C fixed in the square wheel, the graph looks like this (not to scale) (see **fig. 1 [Wilcock]**). The length EB (not to scale) is the arc length from A to B . The length BD (not to scale) is the arc length from B to the y -intercept at $(0, -1)$, which can be found as $\sinh(b)$ by using the arc length calculation. The length $CD = 1$. Thus, by the Pythagorean theorem and the hyperbolic identity $\cosh^2(x) - \sinh^2(x) = 1$, we have $CB = \cosh(b)$. But the coordinates of point B are $(b, -\cosh(b))$, so point C lies on the x -axis.

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to mt@nctm.org. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

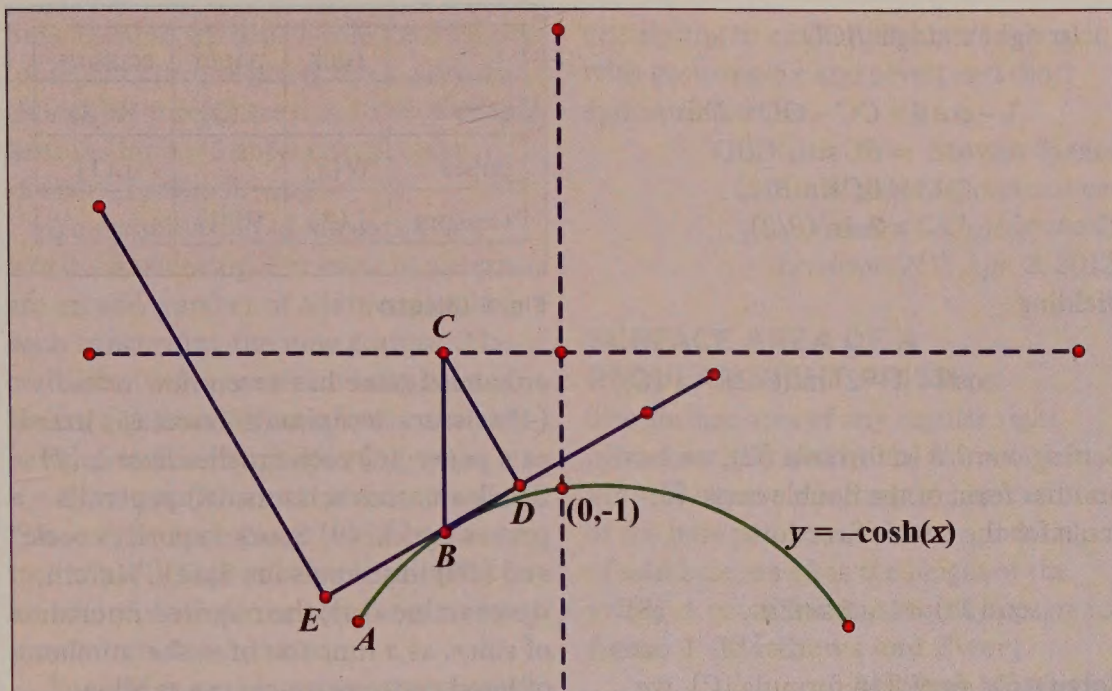


Fig. 1 (Wilcock)

Hence, the axle of the tricycle rides at the x -axis as the tricycle traverses the catenary.

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THE EQUATION OF A CIRCLE

A standard problem in analytic geometry is to find an equation of the circle given the endpoints of a diameter, (x_1, y_1) and (x_2, y_2) . The standard solution is to find the coordinates of the center of the circle and then find the radius and write the equation.

Instead, consider that the segment joining a third point (x, y) on the circle and one endpoint is perpendicular to the segment joining that same third point (x, y) to the other endpoint. The product of the slopes of the two segments is -1 . Thus,

$$\left(\frac{y - y_1}{x - x_1} \right) \cdot \left(\frac{y - y_2}{x - x_2} \right) = -1$$

is the equation of the circle.

For example, if the diameter's endpoints are $(-5, 1)$ and $(3, 4)$, we have

$$\left(\frac{y - 1}{x + 5} \right) \cdot \left(\frac{y - 4}{x - 3} \right) = -1$$

$$\frac{y^2 - 5y + 4}{x^2 + 2x - 15} = -1.$$

So $y^2 - 5y + 4 = -(x^2 + 2x - 15)$, which

can be further simplified to $x^2 + y^2 + 2x - 5y - 11 = 0$.

Although this procedure might not be new, it is underpublicized. Sometimes the general equation is presented as $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, but this form conceals the genesis of the equation.

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SEVEN TRIGONOMETRY FORMULAS FROM ONE GRAPH

Over the past few years, while teaching trigonometry in my honors precalculus and precalculus courses, I found that some trigonometry formulas—especially the double-angle and half-angle formulas—are difficult for my students to derive, understand, and remember. They have no idea how to obtain such formulas as

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin \theta}{1 + \cos \theta} \text{ and } \tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{\sin \theta}$$

because the intermediate steps of their derivation require some algebraic manipulations and double-angle formulas. To help students master these formulas, I start with a graph of a unit circle. From triangles in the unit circle, seven trigonometry formulas can be directly obtained or easily derived (see **fig. 1 [Tu]**).

In the unit circle O , $\overline{AC}$ is the diameter. B is a point on the unit circle, and $\overline{BD}$ is perpendicular to $\overline{AC}$ at D . Assume that $m\angle BOC = \theta$. From geometry, we know that the measure of an inscribed angle is half that of the central angle for the same intercepted arc. Therefore, $m\angle ABC = 90^\circ$, and $m\angle BAC = \theta/2$. Both $\angle BAC$ and $\angle CBD$ are complementary

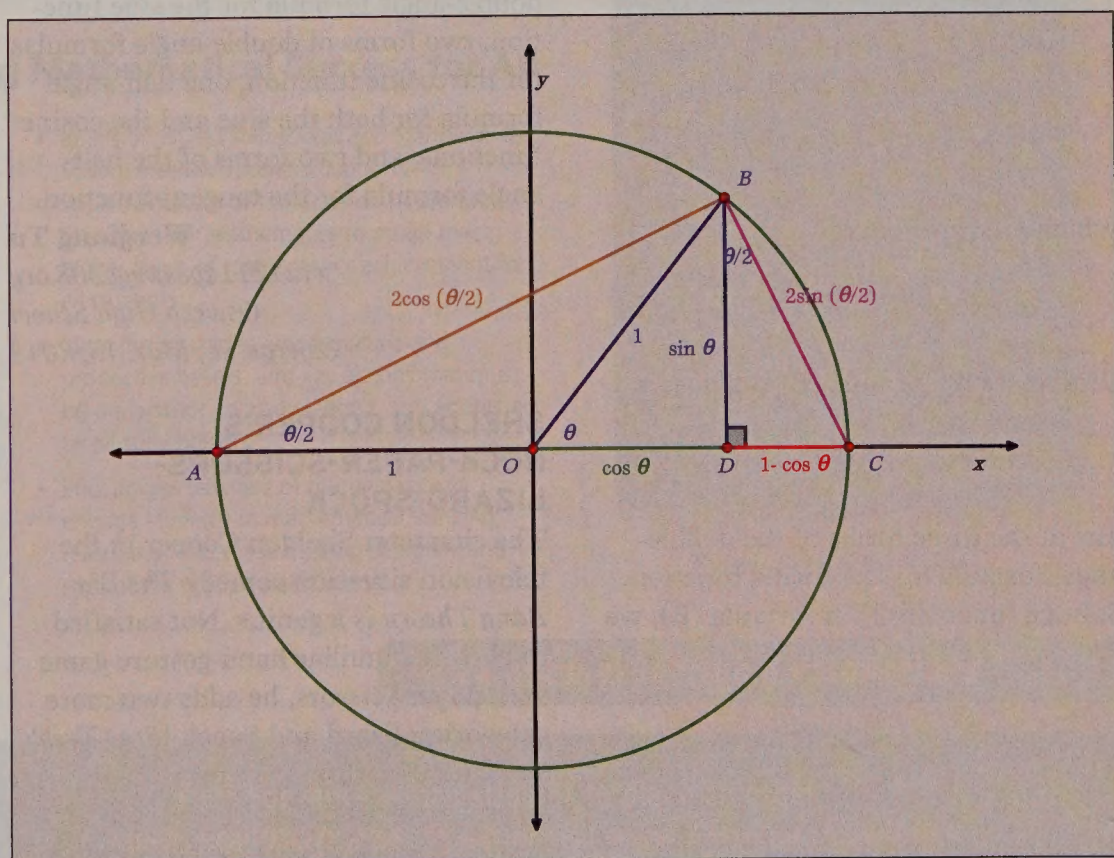


Fig. 1 (Tu)

to the same angle, $\angle ACB$, so $m\angle CBD = \theta/2$.

In right triangles OBD and ABC , we see that

$$\begin{aligned} BD &= OB \sin \angle BOC = 1 \cdot \sin \theta = \sin \theta; \\ OD &= OB \cos \angle BOC = 1 \cdot \cos \theta = \cos \theta; \\ AB &= AC \cos \angle BAC = 2 \cos(\theta/2); \text{ and} \\ BC &= AC \sin \angle BAC = 2 \sin(\theta/2). \end{aligned}$$

In right triangle BAD ,

$$\begin{aligned} \sin \theta &= BD = AB \sin \angle BAC \\ &= 2 \sin(\theta/2) \cos(\theta/2). \end{aligned} \quad (A)$$

If we set $\alpha = \theta/2$ in formula (A), then it becomes

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha, \quad (1)$$

which is the double-angle formula for the sine function.

Still in right triangle BAD ,

$$\begin{aligned} \tan\left(\frac{\theta}{2}\right) &= \tan \angle BAD = \frac{BD}{AD} \\ &= \frac{BD}{AO + OD} = \frac{\sin \theta}{1 + \cos \theta}; \end{aligned} \quad (2)$$

this is one of the half-angle formulas of the tangent function.

Now,

$$\begin{aligned} 1 + \cos \theta &= AD = AB \cos \angle BAD \\ &= AB \cos(\theta/2) \\ &= 2 \cos^2(\theta/2), \end{aligned}$$

which is equivalent to

$$\cos \theta = 2 \cos^2(\theta/2) - 1. \quad (B)$$

With $\alpha = \theta/2$, formula (B) becomes

$$\cos(2\alpha) = 2 \cos^2 \alpha - 1, \quad (3)$$

one of the three forms of the double-angle formula for the cosine function. Solving for $\cos(\theta/2)$ in formula (B), we get

$$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}, \quad (4)$$

the half-angle formula for the cosine function.

In right triangle BCD ,

$$\begin{aligned} 1 - \cos \theta &= OC - OD = DC \\ &= BC \sin \angle CBD \\ &= BC \sin(\theta/2) \\ &= 2 \sin^2(\theta/2), \end{aligned}$$

yielding

$$\cos \theta = 1 - 2 \sin^2(\theta/2). \quad (C)$$

Setting $\alpha = \theta/2$ in formula (C), we have another form of the double-angle formula for the cosine function:

$$\cos(2\alpha) = 1 - 2 \sin^2 \alpha. \quad (5)$$

Solving for $\sin \theta/2$ in formula (C), we obtain

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \quad (6)$$

the half-angle formula for the sine function. Finally,

$$\tan\left(\frac{\theta}{2}\right) = \tan \angle CBD = \frac{DC}{BD} = \frac{1 - \cos \theta}{\sin \theta}, \quad (7)$$

which gives another half-angle formula for the tangent function.

Therefore, this graph prompts one double-angle formula for the sine function, two forms of double-angle formulas for the cosine function, one half-angle formula for both the sine and the cosine functions, and two forms of the half-angle formula for the tangent function.

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Oswego, IL, Mar. 15, 2013

SHELDON COOPER'S ROCK-PAPER-SCISSORS- LIZARD-SPOCK

The character Sheldon Cooper in the television situation comedy *The Big Bang Theory* is a genius. Not satisfied to play the familiar hand-gesture game rock-paper-scissors, he adds two more categories: lizard and Spock (*Star Trek's* Vulcan). The three general rules for winning a skirmish are (1) scissors cuts paper; (2) rock crushes scissors; and (3) paper covers rock. Sheldon's

	rock	paper	scissors
rock		L(3)	W(2)
paper	W(3)		L(1)
scissors	L(2)	W(1)	

Fig. 1 (Siegel)

enhanced game has seven new rules:

(4) scissors decapitates lizard; (5) lizard eats paper; (6) rock crushes lizard; (7) Spock smashes scissors; (8) paper disproves Spock; (9) Spock vaporizes rock; and (10) lizard poisons Spock. Herein, we examine $r(n)$, the required number of rules, as a function of n , the number of hand gestures we choose to allow. The rules are meant to be chosen so that each gesture has an equal chance of winning.

In the familiar game, $n = 3$ and $r = 3$. Each "hand throw" beats one and loses to one of the other two. Therefore, each has an equal chance of winning. In **figure 1 (Siegel)**, the top row summarizes that rock loses to paper (according to rule 3) but that rock wins over scissors (according to rule 2).

Adding lizard to the mix ushers in the creation of rules 4, 5, and 6 because lizard must interact with rock, scissors, and paper. Unfortunately, the game becomes more complicated because, with these first six rules, scissors and rock each have two ways to win and only one way to lose, whereas paper and lizard each have one way to win and two ways to lose. Strangely, the strategy becomes easier when we add Spock. His interaction with the other four gestures launches rules 7 through 10. Now each gesture has two chances to win and two to lose. No throw has an advantage, so we merely try not to be predictable as we throw each 20 percent of the time.

Now let us address $r(n)$. We have seen that $r(3) = 3$, $r(4) = 6$, and $r(5) = 10$. For $n > 3$, $r(n) = r(n-1) + n-1$ because, when we add the n th gesture, we need $n-1$ more rules to govern how the n th gesture must interact with each of the first $n-1$ gestures. Thus, $r(n)$ is the sum of the first $(n-1)$ integers, which is $(n-1)n/2$. It is no coincidence that this sum is precisely $C(n, 2)$, the number of combinations of n things taken two at a

time, because we need a rule for each pair (of distinct throws) from the n gestures. (Matching throws are ties.) The proposed gestures for $n = 5$ appear on Sheldon Cooper's Facebook® page.

An examination of n 's parity is worth considering. For even n , we create an odd number of additional rules, each concerning the new gesture. The additional rules create an imbalance for each gesture. (To maintain as much balance as possible, beginning with $n = 4$, we create rules that keep the "losses" for each gesture within one of its "wins.") This imbalance provides an incentive to throw certain gestures more often than others.

For odd n , we create an even number, $n - 1$, of new rules. They concomitantly repair the imbalance of wins to losses for the prior gestures and balance the wins and losses for the new gesture. Here, we throw each gesture one n th of the time. A creative reader may more easily understand this process by

attempting to extend Sheldon's scenario with gestures six and seven and the appropriate rules.

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SURFACE AREA OF A REGULAR RIGHT PRISM

The surface area of any regular right prism can be found by the formula $S = p(a + h)$, where p is the perimeter of one of the base polygons, a is the apothem of said base, and h is the height of the prism. A proof without words appears in figure 1 (Matthews and Ewer).

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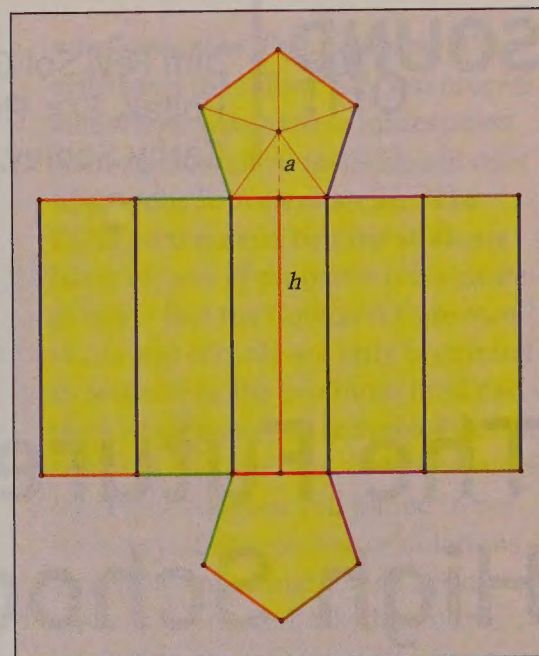


Fig. 1 (Matthews and Ewer)

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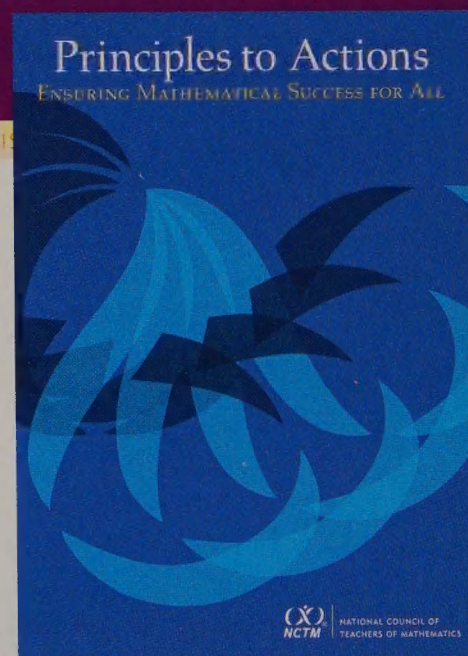
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Jim Fey, Sol Garfunkel, Diane Briars, Andy Isaacs, Henry Pollak, Eric Robinson, Richard Scheaffer, Alan Schoenfeld, Cathy Seeley, Dan Teague, and Zalman Usiskin

The Future of High School Mathematics

Results from the 2012 Program for International Student Assessment (PISA), released on December 3, 2013, showed once again that U.S. high school students are only in the middle of the pack when it comes to science, mathematics, and literacy achievement (OECD 2013). The findings quickly elicited an outburst of public hand wringing, criticism of U.S. schools and their teachers, and calls to emulate the curriculum and teaching practices of high-achieving countries. Then, quite predictably, there were a variety of explanations as to why we cannot import the policies and practices of other quite different countries. Instead, policymakers and pundits with little expertise in mathematics or experience in mathematics education urged schools to redouble efforts along lines that have been largely ineffective for the past decade and are not common in any high-performing country—a regimen of extensive standardized testing with mostly punitive consequences for schools and for teachers who fail to make adequate yearly progress. Public attention to the challenge of international competition soon faded, and we will hear little about the meaning of PISA results until the next wake-up call arrives.

WHAT MIGHT HAPPEN IF WE TRIED SOMETHING DIFFERENT?

Countries that have made real progress in their performance on international

assessments share several characteristics. First and foremost is broad agreement on the goals of education and sustained commitment to change over time. In the United States, there has been steady improvement in student mathematics performance at the elementary school and middle school levels on the National Assessment of Educational Progress (NAEP) and some improvement in results on Advanced Placement and college entrance examinations over the past two decades—a period when efforts have been guided by the National Council of Teachers of Mathematics (NCTM) Standards for curriculum, evaluation, teaching, and assessment.

Over the past three years, forty-five of the fifty U.S. states have been engaged in an effort to implement the Common Core State Standards (CCSS) for Mathematics and English Language Arts. With respect to mathematics, those standards, prepared under the aegis of the National Governors Association with generous private financial support, in many ways extend key ideas in the earlier NCTM Standards (NCTM 2000). Despite understandable controversy about particulars of the CCSS and the processes by which they were developed and the processes by which states were induced to adopt them, the Common Core standards provide a useful framework for further efforts, *provided they are viewed as a living document to be modified as recommended by experience.*

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WHAT SHOULD STUDENTS, TEACHERS, PARENTS, AND POLICYMAKERS LOOK FOR IN THE EMERGING REFORM OF HIGH SCHOOL MATHEMATICS?

From our perspective—as mathematicians, teachers, statisticians, teacher educators, and curriculum developers with extensive experience in school mathematics innovation—there are five key elements of the Common Core program that provide a basis for productive change in U.S. high school mathematics:

▪ **Comprehensive and Integrated Curriculum**—The traditional American high school mathematics curriculum consists of two yearlong courses in algebra and a one-year course in geometry. The CCSS for mathematics retain essential elements of those topics, but they also prescribe attention to important concepts and skills in statistics, probability, and discrete mathematics that are now fundamental in computer, management, and social sciences. The Common Core guidelines also describe an attractive integrated curriculum option—suggested by the common practice in other countries of addressing each mathematical content strand in each school year. That international curriculum design helps students learn and use the productive connections among algebra, geometry, probability, statistics, and discrete mathematics.

A broad and integrated vision of high school mathematics would serve our students better than the narrow and compartmentalized structure of traditional programs.

▪ **Mathematical Habits of Mind**—For most people the phrase “do the math” means to follow standard algorithms for calculation with whole numbers, fractions, decimals, and the symbolic expressions of algebra. But productive quantitative thinking also requires understanding and skill in mathematical “sense making” (Martin et al. 2009) and use of what the Common Core Standards call *mathematical practices*. To apply mathematical concepts and methods effectively in the kind of realistic problem-solving

and decision-making tasks that PISA assessments highlight, students need to develop the habits of (1) analyzing complex problems and persevering to solve them; (2) constructing arguments and critiquing the reasoning of others; (3) using mathematical models to represent and reason about the structure in problem situations; and (4) communicating results of their thinking in clear and precise language.

Developing important mathematical habits of mind, especially the process of mathematical modeling that is required to solve significant real-world problems, should become a central goal of high school instruction.

▪ **Balanced Attention to Technique, Understanding, and Applications**—One of the most common student beliefs about mathematics is that what students are asked to learn is not supposed to make sense and bears little relationship to the reasoning required by everyday life. Those views are expressed well in the whimsical rhyme about division of common fractions, “Yours is not to reason why; just invert and multiply,” and the common student question, “When will I ever use this stuff?” Unfortunately, many teachers encourage those beliefs about mathematics learning by suggesting that understanding and application of mathematical ideas and methods can occur only after rote mastery of technical skills.

Findings of cognitive and curriculum design research over the past two decades challenge such conventional beliefs and common practices. Curricula and teaching that engage students in collaborative exploration of realistic problems have been shown to be effective in developing student mathematical understanding, skills, and problem solving simultaneously. These problem-based approaches in the classroom also develop students’ disposition to use mathematics as a reasoning tool outside school.

Improved performance on international assessments like PISA are likely to result from moves toward curricula and teaching methods that balance and integrate mathematical techniques, understanding, and applications.

▪ **Information Technologies**—Powerful tools that allow users to process visual and quantitative information with mathematical methods are now ubiquitous in American life. The CCSS recommend helping students learn to “use appropriate tools strategically.” But the Common Core standards and schools are only beginning to respond to the profound implications of information technology for teaching and learning. If it is possible to simply ask your cell phone to perform any of the routine calculations taught in traditional school arithmetic, algebra, and calculus courses, what kind of mathematical learning remains essential? If those same tools can be applied to support student-centered exploration of mathematical ideas, how will the new learning options change traditional roles of teachers and students in the mathematics classroom and raise expectations for the mathematical challenges that students can tackle?

Intelligent response to the challenges and opportunities presented by information technologies will require creative research and development efforts and the courage to make significant changes in traditional practices.

• **Probing and Useful Assessments**—One of the clearest findings of educational research is the truism that what gets tested gets taught. PISA is not a perfect or complete measure of high school student achievement. Neither are the TIMMS international assessments, the NAEP tests, the SAT and ACT college entrance exams, college placement exams, or, quite likely, the coming assessments attached to the Common Core State Standards (Larson and Leinwand 2013).

Some would respond to the inadequacy of current assessment tools by sharply curtailing high-stakes standardized testing; others would increase the testing and raise the consequences for students and schools who perform poorly. It is almost certainly true that the best course lies somewhere between those extremes.

We need new and better tools for assessing student learning—especially in the areas of mathematical modeling, problem solving, and quantitative reasoning. Then we need to see that those assessments are used in constructive ways to help teachers improve instruction and to inform educational policy decisions.

WE NEED A CHANGE IN THE CULTURE

Mathematics educators who have been active in reform efforts for the past two decades will probably point out that our ideas for the future of high school mathematics have been expressed in earlier policy documents and curriculum projects inspired by the NCTM Standards. But it seems safe to say that we have a long way to go before those recommendations are common practice in most U.S. high schools. The force of tradition in education makes its aims and practices consistently and powerfully resistant to change.

Mathematics teachers, curriculum developers, and researchers have made significant progress in developing and

testing efficacy of comprehensive and integrated curriculum structures; of problem-based teaching that develops student skill, understanding, and problem solving; of strategies for teaching mathematical practices like modeling complex realistic situations; of the use of technology for doing and teaching mathematics; and of assessment for breadth and depth of learning. However, *if that progress is to be sustained and broadly implemented, we need to accelerate the pace of innovation.*

To apply national resources to that effort in optimal ways, we must regularly remind ourselves that education is a community responsibility that aims to develop the abilities and interests of all students, not a contest with winners and losers. We need to work together to develop progressive goals for school mathematics and high-quality instructional resources. Most important, we need to change the tenor of public and professional discourse about mathematics education. We need to dial down the acrimonious policy arguments and relentless criticism of schools and teachers. We

in the profession need to be articulate and persistent in making the case that teaching is one of the most important and demanding tasks for adults in our society and that teachers deserve our wholehearted encouragement and support.

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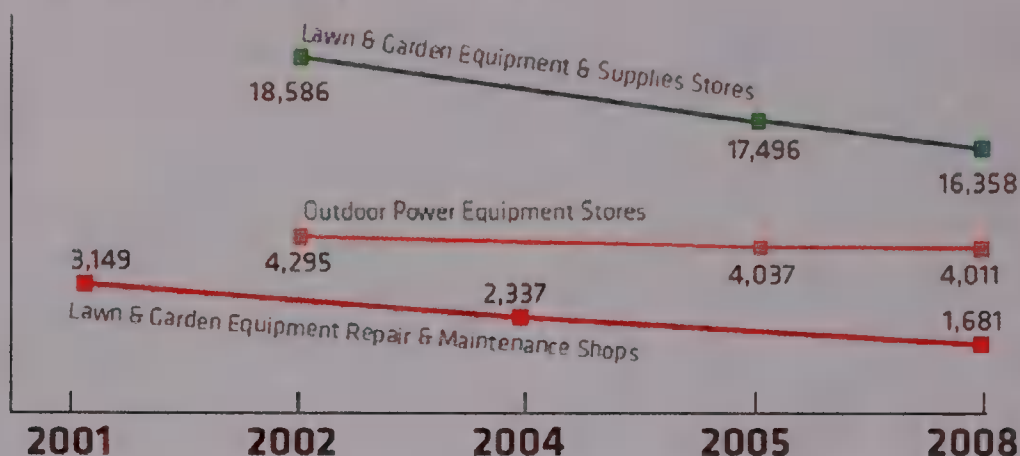
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Lawn and Garden

Number of U.S. Businesses



NOTE: Outdoor Power Equipment stores are a subset of Lawn & Garden Equipment & Supplies stores. Numbers based on U.S. Census Bureau data.

Source: Gregg Wartgow, "Critical Condition," *Yard and Garden*, vol. 34, no. 4, pp. 8–12

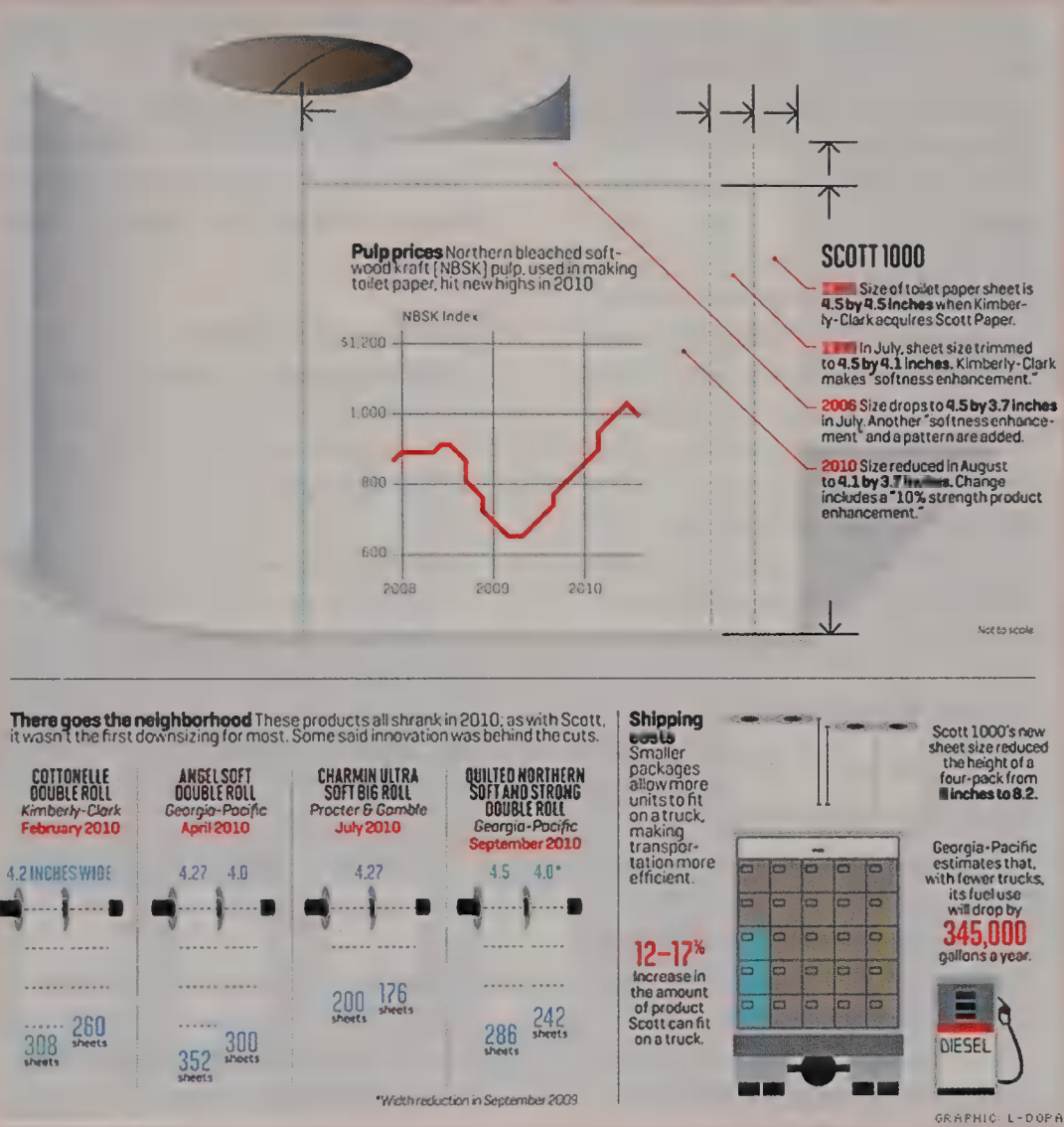
- In what year is the number of Outdoor Power Equipment stores the greatest percentage of Lawn & Garden Equipment & Supplies stores?
- Are the three graphs depicted in the illustration linear? Justify your answer.
- Examine the horizontal scale of the graph and explain why the scale could misrepresent the data.
- Examine the vertical scale of the graph and explain why the scale could misrepresent the data.
- Use the graph of Lawn & Garden Equipment Repair & Maintenance Shops to answer the following questions:
 - Are more stores opening or closing? Explain your answer.
 - During which time span, 2001–2004 or 2004–2008, was the number of stores decreasing more rapidly?
- Using only the data from 2001 to 2004, estimate how many stores were open in 2002.
- There are 2,381 establishments for the year 2002. (Go to <http://factfinder2.census.gov/faces/tableservices/jsf/pages/productview.xhtml?src=bkmk>.) How does the number of stores (establishments) compare with your answer for the year 2002? Explain any differences between your answer to part 5(c) and the value in the table.
- Redraw the given graph with an appropriate horizontal and vertical scale. Extend the graph for each type of store until the year 2050. Explain your choices for the scale and the resulting graph.

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Pulp Paper



Source: <http://features.blogs.fortune.cnn.com/2010/11/09/why-your-toilet-paper-is-shrinking/>

- (a) Use the graph to give approximate intervals during which pulp prices decreased.
- (b) Use the graph to give approximate intervals during which pulp prices increased.
- (c) Give approximate values for the average slope of the graph over the interval from June 2008 to June 2009. Be sure to include units in your answer.

- What is the approximate percentage increase in the price of pulp from its lowest price to its highest price?

Table 1 ("Pulp Paper")				
	Cottonelle®	Angel Soft®	Charmin®	Quilted Northern®
Price per roll	\$1.40	\$0.81	\$0.75	\$0.95
January 2010 unit cost				
December 2010 unit cost				

- In August 2010, by how much did the area of a sheet of Scott® 1000 toilet paper decrease?
- When did the area of a sheet of Scott 1000 toilet paper have the largest percentage decrease?
- If the diameter of a roll of Scott 1000 toilet paper is 6 inches, how much did the volume of a roll decrease from 2006 to 2010?
- How many feet taller was a stack of 50 rolls of Scott 1000 toilet paper in 1999 than in 2010?
- If the average cost of a gallon of diesel gasoline in 2010 was \$4.10, how much money did Georgia-Pacific save in 2010?
- Having reduced the size of its four-pack, the Scott company can increase the amount of product it can fit on a truck. Estimate the number of packages of toilet paper a truck can hold if it previously carried 9,600 packages.
- For the graphs under the heading "There goes the neighborhood," what scale is used to represent the number of sheets per roll?
- If the length of a sheet of Angel Soft toilet paper is 3.7 in., how much shorter is a roll of Angel Soft after the size reduction in April 2010?
- The cost of one roll of toilet paper for the four brands is listed below. Find the unit cost per 100 sheets before and after the products shrank in 2010 (use **table 1** ["Pulp Paper"]).

"Lawn and Garden" answers

1. The percentage is largest in the year 2008. The comparison for the three years follows:

2002: 4295/18586 ≈ 23.1%
2005: 4037/17496 ≈ 23.1%
2008: 4011/16385 ≈ 24.5%

2. None of the graphs is linear. By calculating the slope between the given points, we can show that they are not the same.

Lawn & Garden Equipment & Supplies Stores (green line):

(18586 - 17496) / (2002 - 2005) ≈ -357.3
(17496 - 16358) / (2005 - 2008) ≈ -379.3

Outdoor Power Equipment Stores (yellow line):

(4295 - 4037) / (2002 - 2005) = -86
(4037 - 4011) / (2005 - 2008) ≈ -8.7

Lawn & Garden Equipment Repair and Maintenance Shops (red line):

(3149 - 2337) / (2001 - 2004) ≈ -270.7
(2337 - 1681) / (2004 - 2008) = -164

3. The horizontal scale is not consistent between the years. Although the spacing between tick marks is the same, the time intervals are different, representing differences of 1, 2, and 3 years.

4. No vertical scale is marked on the axis. When we examine the 2008 data, the vertical distances between the plotted points look similar, but the actual differences between the y-coordinates are significantly different (12,347 compared with 2,330).

5. (a) That the line representing the repair and maintenance shops has a negative slope means that the number of shops is decreasing; thus, more stores are closing.

(b) On the basis of the answers to question 2, the number of stores decreased more rapidly in the time span 2001–2004.

(c) The slope between points at 2001 and 2004 is approximately –270.7 stores/year. So we expect 3149 – 270.7 = 2878.3, or 2878 stores were open in 2002.

(d) The number of Lawn & Garden Equipment Repair & Maintenance Shops in 2002, according to the U.S. Census (see table 1 ["Lawn and Garden"]), is 2381, many fewer than the 2878 predicted in part 5(c). One possible explanation is that because the data are not linear, as was determined in question 2, using slope to predict the number of stores does not produce an accurate answer. That the estimate in part 5(c) is too high is consistent with a graph that is concave up.

6. Student graphs will vary. Student graphs should include both a horizontal and a vertical scale with equal increments. Two of the graphs should "level off" toward the year 2050, as an exponential decay model would, but the graph representing the Lawn & Garden Equipment & Supplies Stores may be concave down.

"Pulp Paper" answers

1. (a) Pulp prices decreased from mid-year 2008 to early in 2009, from approximately July 2008 to February 2009.

(b) Pulp prices increased from early 2009 to mid-year 2010, from approximately March 2009 to May 2010.

(c) From June 2008 to June 2009, the slope is

(700 - 900) / (2009 - 2008) ≈ -\$200 / year.

2. From the graph, the highest price is approximately \$1050, whereas the lowest price is approximately \$650. The percentage increase in the price of pulp from the lowest price to the highest is approximately 62%:

Percentage increase = (highest price - lowest price) / lowest price * 100%
= (1050 - 650) / 650 ≈ 62%

Table 1 ("Lawn and Garden")

Table with 10 columns: Geographic area name, 2002 North American Industry Classification System (NAICS) code code, Meaning of 2002 North American Industry Classification System (NAICS) code code, Meaning of Type of operation or tax status code code, Year code, Number of establishments, Receipts/Revenue (\$1,000), Annual payroll (\$1,000), First-quarter payroll (\$1,000), Number of paid employees for pay period including March 12. Rows include United States (2007) and United States (2002).

Source: U.S. Census Bureau, 2007 Economic Census

Table 2 ("Pulp Paper")

	Cottonelle	Angel Soft	Charmin	Quilted Northern
Price per roll in dollars	1.40	0.81	0.75	0.95
January 2010 unit cost (in dollars per 100 sheets)	(100)(1.40/308) ≈ 0.45	(100)(0.81/352) ≈ 0.23	(100)(.75/200) ≈ 0.38	(100)(.95/286) ≈ 0.33
December 2010 unit cost (in dollars per 100 sheets)	(100)(1.40/260) ≈ 0.54	(100)(0.81/300) ≈ 0.27	(100)(.75/176) ≈ 0.43	(100)(.95/242) ≈ 0.39

3. The area of a sheet of Scott 1000 toilet paper was 16.65 in.² in 2006 and 15.17 in.² in 2010. The decrease was 1.48 in.².

4. The area of a sheet of Scott 1000 toilet paper had the largest percentage decrease in July 2006:

$$1999: 1.8/20.25 \approx 8.9\%$$

$$2006: 1.8/18.45 \approx 9.8\%$$

$$2010: 1.48/16.65 \approx 8.9\%$$

5. The width of a sheet (or height in the figure) was 4.5 in. in 2006 and 4.1 in. in 2010. The radius of the roll remained constant at 3 in. The change in volume of the cylindrical roll was

$$\begin{aligned} V_{\text{decrease}} &= V_{\text{year 2006}} - V_{\text{year 2010}} \\ &= \pi \cdot 3^2 \cdot 4.5 - \pi \cdot 3^2 \cdot 4.1 \approx 11.3 \text{ in.}^3. \end{aligned}$$

6. The width of the toilet paper in 1999 was 4.5 in. and in 2010 was 4.1 in. The height difference is

$$\begin{aligned} h_{\text{difference}} &= h_{\text{year 1999}} - h_{\text{year 2010}} \\ &= 50 \cdot 4.5 - 50 \cdot 4.1 = 20 \text{ in.} \approx 1.7 \text{ ft.} \end{aligned}$$

7. Since the amount of fuel dropped by 345,000 gallons in a year, Georgia-Pacific saved \$1,414,500:

$$345,000 \text{ gallons} \cdot \frac{\$4.10}{\text{gallon}} = \$1,414,500$$

8. The increase is 12% to 17%. The 12% increase can be computed as $9,600 \cdot 1.12 = 10,752$; the 17% increase can be computed as $9,600 \cdot 1.17 = 11,232$. Thus, the truck can carry between 10,752 and 11,232 packages.

9. The scale is 100 sheets per square. If we examine the Angel Soft Double Roll, the right image shows three full squares, representing 300 sheets in the roll. If we examine Charmin Ultra Soft Big Roll, the left image shows two full squares, representing 200 sheets in the roll.

10. The roll is 192.4 in. shorter. Angel Soft shortened its rolls of toilet paper from 352 to 300 sheets. So each roll would then be $52 \cdot 3.7 = 192.4$ in. shorter.

11. See **table 2 ("Pulp Paper")**.



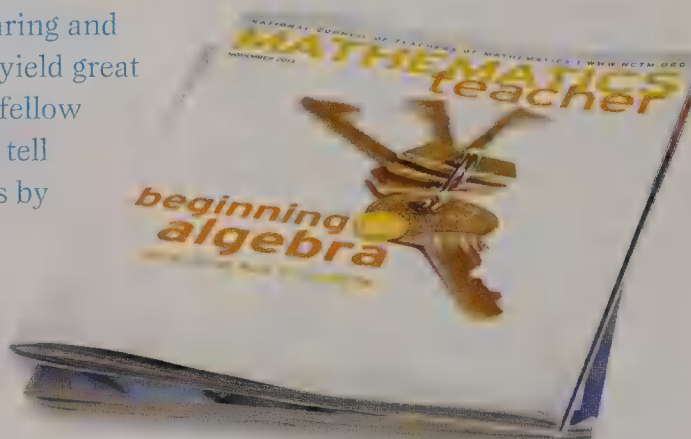
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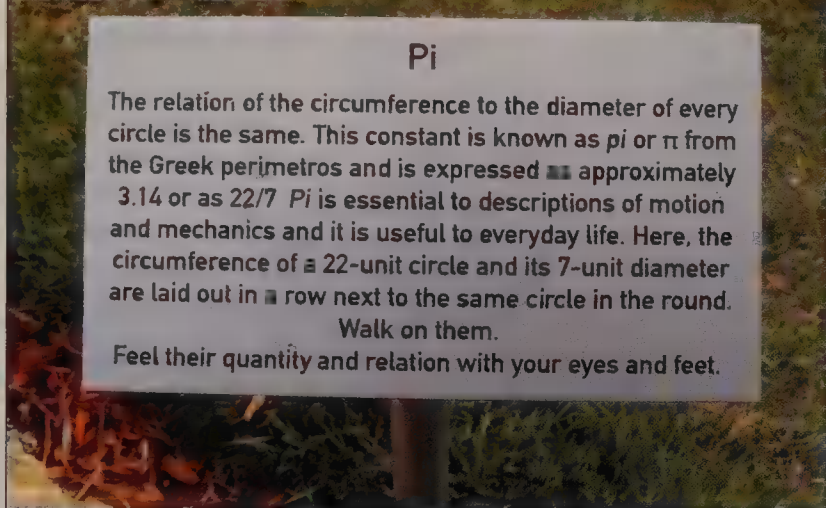
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Pi All Month

Photograph 1
Pi plane, East Garden,
Salvador Dali Museum



Photograph 2
Pi walkway



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The East Garden of the Salvador Dali Museum in St. Petersburg, Florida, pays homage to the mathematical concept of π with a walkway that illustrates the relationship between a circle's circumference and its diameter. **Photograph 1** shows a plaque that describes this relationship, and **photograph 2** gives an overview of the walkway that is

described. Students often learn about this relationship by measuring the circumference and diameter of circles of various different sizes and evaluating the ratio of the circumference to the diameter of each. Few, however, experience this relation by walking it.

1. For each circle shown in **figure 1**, a diameter is drawn. Use a string to measure the diameter and show that the distance around the circle is approximately 3 times the measure of the

diameter. Pi has been known for almost 4000 years, according to http://www.exploratorium.edu/pi/history_of_pi/.

2. Archimedes (ca. 250 BCE) used inscribed and circumscribed polygons with increasing numbers of sides to find a better approximation of π than 3. He used Euclidean geometry principles and computations done by hand to measure the perimeter of these polygons, thereby arriving at the inequality

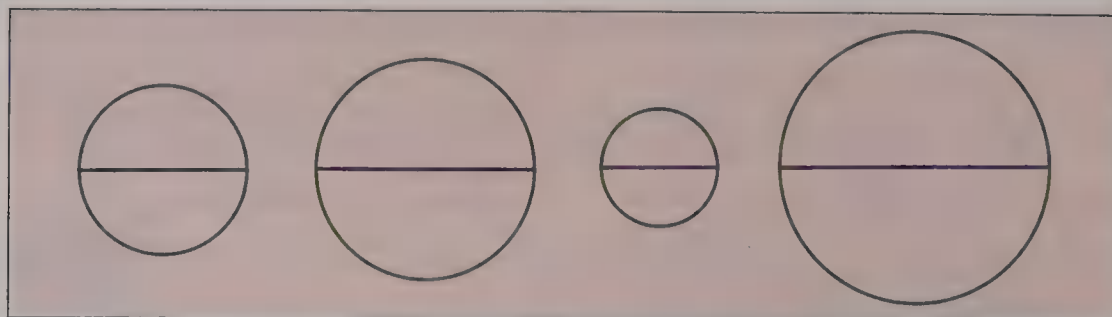


Fig. 1 The ratio of the circumference to the diameter should be the same for all these circles.

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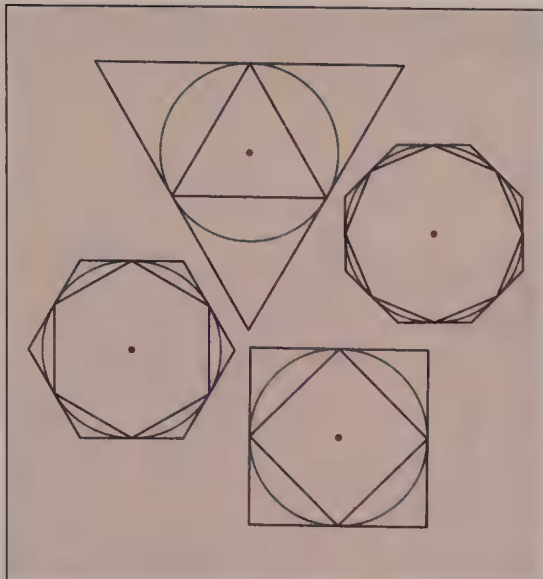


Fig. 2 Averaging the perimeters of the inscribed and circumscribed polygons for a circle of radius 1 gives an approximation of π .

$$3\frac{10}{71} < \frac{\text{circumference}}{\text{diameter}} < 3\frac{1}{7}.$$

Let the diameter of each circle shown in **figure 2** be 1 unit.

- Find the perimeter of the inscribed and circumscribed triangles, squares, and hexagons.
- Use trigonometry to find the perimeter of the inscribed and circumscribed octagons.
- Create an inequality that approximates the ratio of the circumference of a circle to its diameter.

Use of the symbol π to represent the ratio of the circumference to the diameter was first introduced by William Jones (1675–1749) and then popularized by Leonhard Euler (1707–83). **Photograph 3** offers a closer look at the stones of the circular walkway. Notice that the 7 stones across the center walkway are tight together, but there is a gap between them and the enclosing circular path.

Figure 3 shows a sketch of this arrangement of stones. **Figure 4** shows a detail of the gap: $\overline{AB}$ represents the outer edge of the rightmost stone in the diameter walkway, and $\overline{CD}$ represents the inner edge of the adjacent stone in the circular path. The circle with center O that



Photograph 3
The circular walkway

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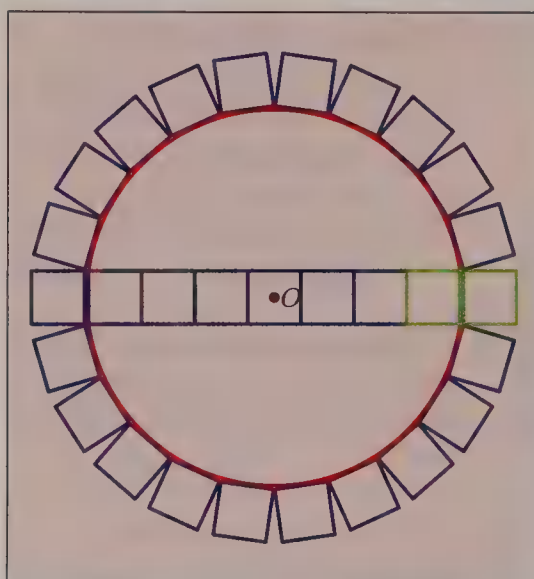


Fig. 3 The gap region between the green tiles is enlarged in **figure 4**.

intersects $\overline{AB}$ has radius OA and tangent $\overline{CD}$.

- Assume that all stones are 1-foot squares.
 - The 7 stones across the middle of the circular walkway represent the number of units in the diameter of the inner circle, and the 22 stones represent the number of units in the circumference. What is the circumference of the circle with radius OA ?
 - With 22 square stones tangent to

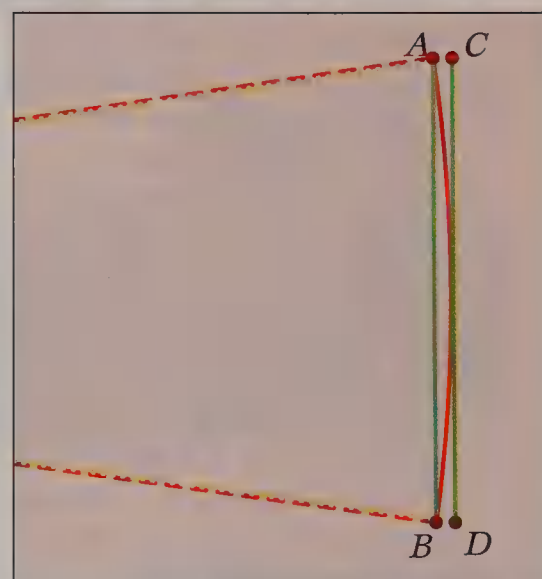


Fig. 4 Chord $\overline{AB}$ intercepts arc $\overline{AB}$; $\overline{CD}$ is a tangent.

the circle, what is the circumference of the circle with radius OC ?

- How big is the gap between the outer stone of the diameter walkway and the adjacent stone of the circular path—that is, the distance between AB and CD ?
- What is the circumference of the smallest circle that can contain the walkway?
- How do these values explain the gaps that exist between the stones on the circular path?

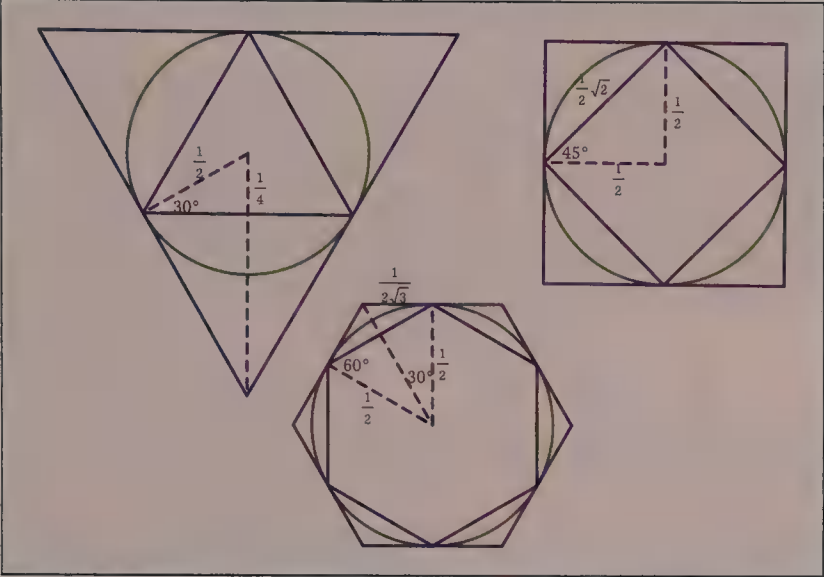


Fig. 5 Right triangles are used to calculate polygon side lengths.

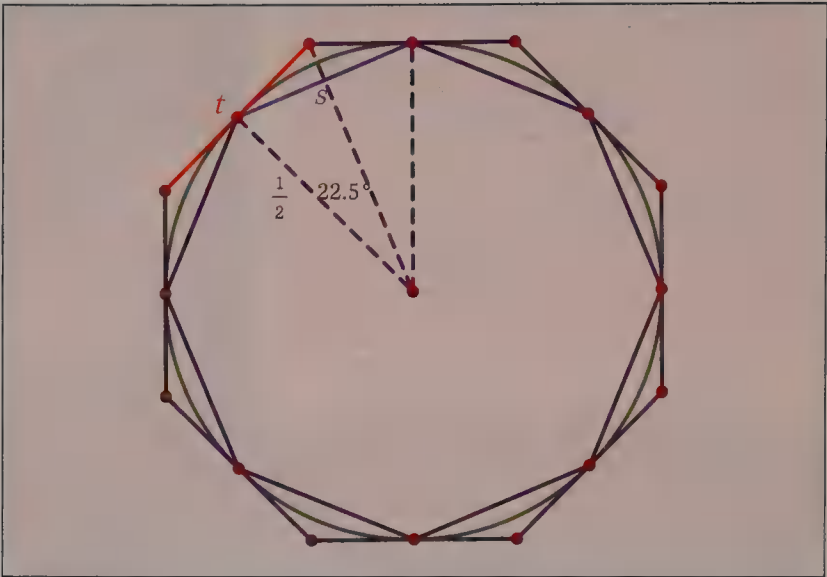


Fig. 6 The central angle is used to calculate octagon lengths.

1. The string around the circle will be a little more than 3 times the length of the string that measures the diameter, no matter what size the circle is.
2. (a) Using 30-60-90° triangle relationships, we can calculate side lengths for the triangles and hexagons. Using 45-45-90° relationships, we can calculate the side lengths for the squares. These values are shown in **figure 5**. The corresponding perimeters, as integer multiples of the side lengths, are shown in **table 1**.
- (b) Each central angle of the octagon is 45°. We form right triangles with 45/2° and label the side of the inscribed octagon, s , and the side of the circumscribed octagon, t (see **fig. 6**). Then $\sin 22.5^\circ = (1/2)s / (1/2) = s$, and $\tan 22.5^\circ = (1/2)t / (1/2) = t$. Therefore, the perimeters, in feet, of the inscribed and circum-

scribed octagons are $8 \sin 22.5^\circ \approx 3.06$ and $8 \tan 22.5^\circ \approx 3.31$, respectively. (The side of an inscribed octagon can also be found using the law of cosines since we are given two radii with length $1/2$ and the included angle of 45° .)

- (c) The octagons approximate the circumference better than the polygons having fewer sides. Therefore, $3.06 < \text{circumference} < 3.31$.
3. (a) The outer edge of the diameter walkway is a chord of the circle with radius OA . A right triangle can be formed with hypotenuse OA and legs $1/2$ and $7/2$. Therefore, the radius OA is $\sqrt{50}/2 = 5\sqrt{2}/2$, and the circumference is $2\pi \cdot 5\sqrt{2}/2 = 5\pi\sqrt{2} \approx 22.21$ ft.
- (b) The radius OC is the hypotenuse of a right triangle whose legs are $1/2$ and OA . Therefore, OC is

$\sqrt{51}/2$, and the circumference is $2\pi \cdot \sqrt{51}/2 = \pi\sqrt{51} \approx 22.44$ ft.

(c) The gap between the stones is equal to the radius OA minus half the length of the 7-stone center walkway. Therefore, the distance is $5\sqrt{2}/2 - 3.5 \approx 0.036$ ft.

- (d) The smallest circle that can contain the walkway must include the stone whose left edge is $\overline{CD}$. The radius of that circle is the hypotenuse of a right triangle with legs $1/2$ and $OA + 1$. Therefore, the radius is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2} + 1\right)^2} \approx 4.563 \text{ ft.}$$

The circumference of this circle is $2\pi \cdot 4.563 \approx 28.67$ ft.

- (e) The stones making up the circular path are contained between an inside circle having circumference approximately 22.44 ft. (found in part 3[b]) and an outside circle whose circumference is approximately 28.67 ft. (found in part 3[d]). The circumferences' increase accounts for the widening gaps between the stones on the circular walkway as we move outward from the center.

Table 1 Lengths (in ft.) of Inscribed and Circumscribed Polygons				
	$S_{\text{inscribed}}$	$S_{\text{circumscribed}}$	$P_{\text{inscribed}}$	$P_{\text{circumscribed}}$
Triangle	$\sqrt{3}/2$	$\sqrt{3}$	$3\sqrt{3}/2$	$3\sqrt{3}$
Square	$\sqrt{2}/2$	1	$2\sqrt{2}$	4
Hexagon	$1/2$	$1/\sqrt{3}$	3	$6/\sqrt{3} = 2\sqrt{3}$



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The practice of problem posing is as important to develop as problem solving. The resulting explorations can be mathematically rich.

A Rationale for Irrationals: An Unintended Exploration of

Nicholas H. Wasserman

Although mathematicians from many early cultures assumed that all positive numbers could be written as a ratio of two natural numbers (i.e., rational numbers), irrational numbers are in fact the solution to many interesting problems in mathematics. The diagonal of a unit square and the circumference of a unit circle, for example, both have irrational lengths. Indeed, irrational numbers can arise from deceptively simple situations. What began as a mathematical task to encourage problem solving unexpectedly led to the irrational number e .

Today, the Common Core State Standards for Mathematics (CCSSI 2010) expect students in as early as eighth grade to be knowledgeable about irrational numbers. In particular, students are to understand infinite decimal expansions and convert between rational numbers' decimal expansions and fractional forms (8.NS.1) and arrive at increasingly better approximations for irrational numbers through iteration (8.NS.2). Later, they are required to learn how to operate with and understand properties of irrational numbers (N-RN.3).

Yet a common tendency in classrooms and on standardized tests is to avoid rational and irrational solutions to problems in favor of integer solutions, which are easier for students to comprehend and check. Without practice and without the mathematical sophistication needed for more rigorous proofs about the irrationality of numbers, students' appreciation of the importance and behavior of irrational numbers may be limited. Nonetheless, activities such as looking at iterations of squared numbers that get closer to 13 (Lewis 2007), constructing the number line (Coffey 2001), and studying the circumference of a circle through regular n -gons as n gets large (Wasserman and Arkan 2011) promote understanding of the infinite behavior of irrational numbers and help develop students' broader conceptions of number.

Common irrational numbers (such as π , square roots, e , and logarithms) as well as their applications in our world are frequently approached from their traditional developments. The irrational number e , for example, is often discussed in the familiar contexts of continuously compounded interest, which results from the fact that



$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e;$$

or, in calculus, as the exponential function with the property that $f'(x) = f(x)$; or as the sum of an infinite series,

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

In contrast with these more traditional developments, this article presents an alternative approach to uncovering the irrational number e . The activity illustrates a “low entry, high ceiling” mathematical task, which could be used in a variety of mathematics courses to build appreciation for and understanding of irrational numbers. What started as a problem-solving activity with in-service K–12 teachers led surprisingly to the irrational solution, e . This article provides an exploration of how the irrational number e emerges from various solution strategies to an unconventional problem and an

example of how problem posing can lead to interesting mathematics.

The term *problem posing* is borrowed from *The Art of Problem Posing* (Brown and Walter 2005) and refers to shifting the mathematical involvement of students to include not just the solving but also the formulation of problems. In particular, Brown and Walter articulate the connection between problem solving and problem posing in mathematics and argue that this interaction potentially deepens understanding and fosters creative inquiry. The “What-If-Not” approach they describe for posing mathematical problems, of which this article provides one example, encourages students to list specific attributes of a problem and then explore the mathematical implications for modifying the problem by removing one or more assumptions.

PARTITIONING FOR MAXIMUM PRODUCT

In a graduate mathematics education course titled Numbers and Operations, in-service K–12 teachers (referred to here as students) were given the following problem-solving task:

Find positive integers whose sum is 2012 and whose product is as large as possible.

A large number (2012 is arbitrary) requires students to use reasoning and problem-solving techniques rather than rely completely on calculators.

Students’ first approach frequently involved splitting 2012 in half and comparing 1006×1006 with other possibilities, such as 1005×1007 (see **fig. 1**). When asked why 1006×1006 must be the maximum, students made comparisons to a square being the maximum-area rectangle with a given perimeter and to a vertex as the maximum of a parabola. Many students were satisfied with their answer; however, the instructor’s probing about their assumption, which rests on splitting into two numbers, quickly prompted students to begin exploring partitions of more than two numbers: “Wait, so we could do multiple numbers, like 1000, two times, and 12 [$1000 + 1000 + 12$].” This partition gives a product ($1000 \times 1000 \times 12$) that is approximately 12 times larger than 1006×1006 . Realizing that the problem allows for such partitions inspired students to explore further. The two methods described here were developed during the course activity.

Method 1: Solve a Simpler Problem

Following one of Pólya’s (1971) problem-solving strategies—“Solve a simpler problem”—some students began to explore the same problem but with smaller numbers. **Figure 2** reveals the work of students who looked at the maximum product for

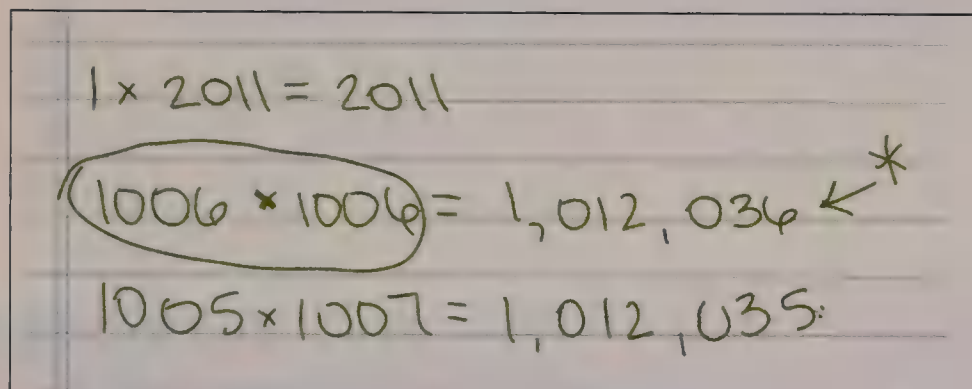


Fig. 1 Students easily found the maximum product for two numbers.

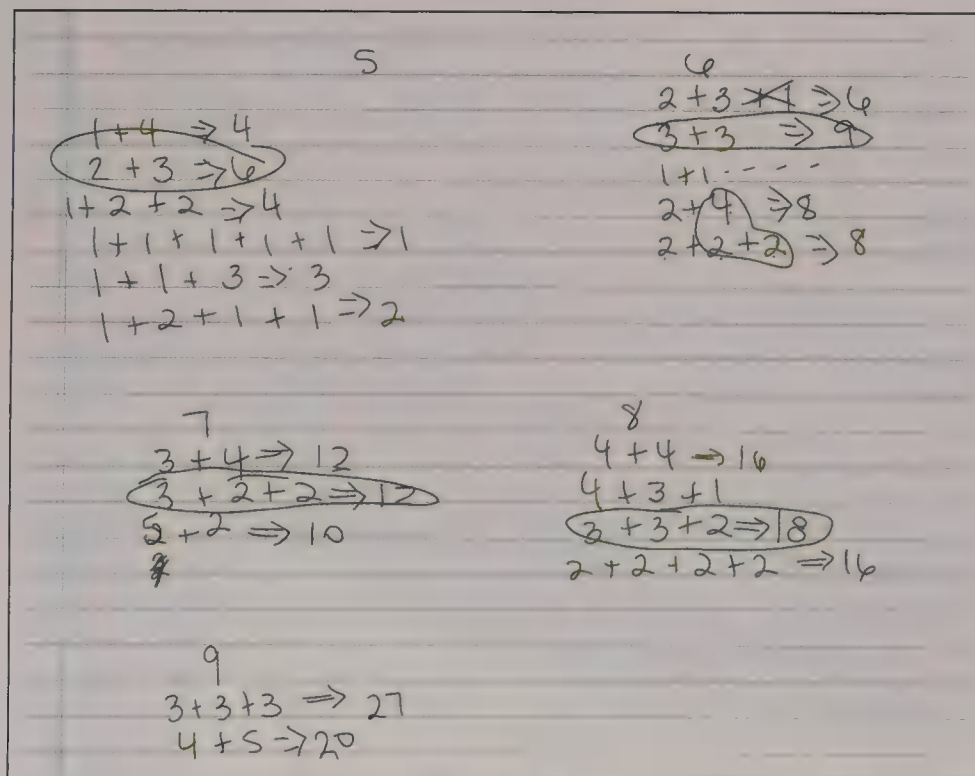


Fig. 2 Some students explored simpler problems.

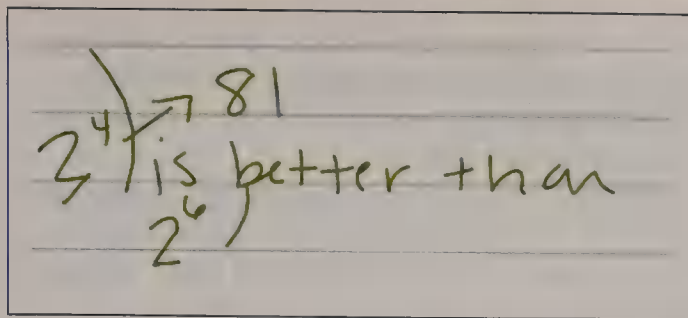


Fig. 3 Students reached a general conclusion about maximum product partitions.

partitions of 5, 6, 7, 8, and 9. (Students quickly realized that partitions using 1 did not increase the product.) Some groups “figured out it has to be combinations of 2s and 3s” but remained unsure about a more general conclusion. Their reasoning fluctuated from trying to have all of one or all of the other (i.e., all 2s or 3s) to having equal quantities of both. Not until students explored the partition of 12 did they generalize an important result (see **fig. 3**): Splitting into 3s provides a larger product because $2 + 2 + 2 + 2 + 2 + 2 = 3 + 3 + 3 + 3$, but $2^6 < 3^4$. Students verified that other examples, including 6, used mostly 3s as well.

The conclusion from these simpler problems was insightful: “So if we went back and broke [2012] into 6s. . . .” This thought helped some students overcome a common misconception in the original example of 2012—that the integer splits all had to be the same number (e.g., all 2s or all 6s). For any sum of 6, splitting into two 3s (and not three 2s) results in the maximum product; therefore, students reasoned that splitting 2012 into three hundred thirty-five 6s, with a remainder of 2, means that the maximum product must incorporate six hundred seventy 3s, with a 2 remaining—that is, the maximum product is $3^{670} \cdot 2$.

Method 2: Reason through Experimentation

Other students did not change the constraints of the original problem (they also used 2012), and, although the approach of exhausting all possibilities leads more toward exhaustion than a solution, a connection to the commutative property of multiplication led to some insight. Students observed that dividing 2012 by 4 could yield either four groups of 503 ($503 + 503 + 503 + 503$) or five hundred three groups of 4 ($4 + 4 + 4 + \dots$). They realized that 4^{503} is significantly larger than 503^4 (see **fig. 4**).

Prompted by the instructor to generalize their conclusions thus far, students responded, “The more amounts we have to multiply by, the more numbers there are in the sum, the larger the [product].” This realization forced some students to conclude (falsely) that the largest exponent, which would come from splitting into groups of 2s, would be the largest product—that is, 2^{1006} .

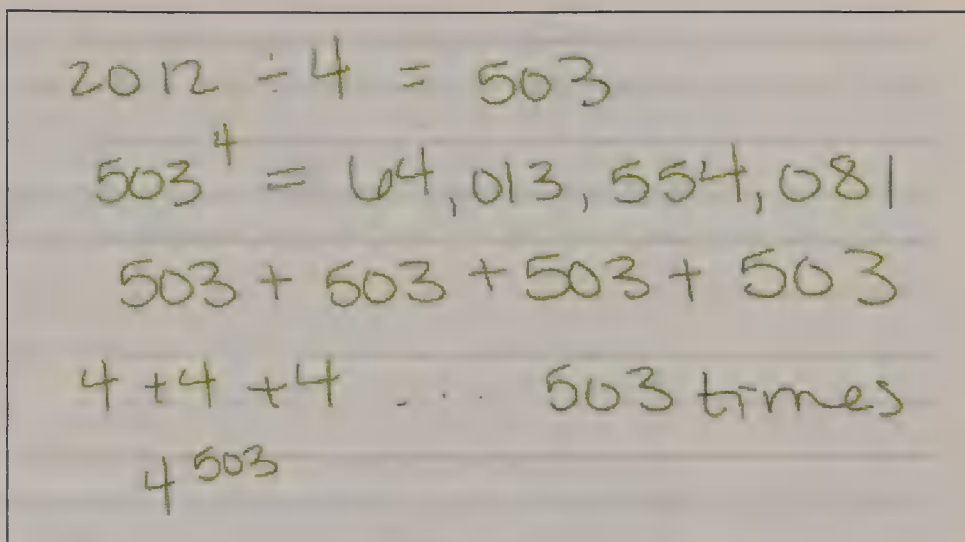


Fig. 4 Students gain insight into the partition.

One group connected this hypothesis with the fact that exponential functions grow more quickly than polynomial functions, implying that larger exponents would have larger products. This conclusion was not unreasonable, but the base of the exponent also matters; the instructor provided a counterexample from another group’s work showing that $2^3 < 3^2$. These students then explored splitting 2012 into different groups but also initially had difficulty incorporating partitions that used different numbers (e.g., not all 2s or all 4s). Using 2012’s prime factorization, they compared the six options (1^{2012} , 2^{1006} , 4^{503} , 503^4 , 1006^2 , and 2012^1) and concluded that both 2^{1006} and 4^{503} were sufficient because they are equivalent.

At this point, the instructor demonstrated, with an example of two hundred one 10s and one 2, how partitions could use different numbers and advised students to look at simpler problems, using an approach similar to method 1. These students likewise arrived at the conclusion that the maximum product partition would use groups of 3: $3^{670} \cdot 2$. (As an extension, students could use logarithms to compare this product with 2^{1006} on a calculator: $670 \cdot \log 3 + \log 2 > 1006 \cdot \log 2$.)

This second approach led to an intriguing question: Why is splitting into groups of 3 better than splitting into groups of 2 in this problem? (Or, in one student’s words, “But are 3s better than 2s?”) From the perspective of exponents, it was unclear why the smaller exponent on 3 in this problem yielded a bigger product than the larger exponent on 2. This question led to an unintended exploration of the irrational number e .

AN UNINTENDED EXPLORATION

A slight modification of the original task generated a new problem-solving adventure.

Find positive *numbers* whose sum is 2012 and whose product is as large as possible.

One word, *integers*, was replaced; this small change removed an assumption about the original problem (that the partition must be an integer partition) and required approaching the task from a different perspective. Essentially, the new question can be read as, “What if the partitions did *not* have to be whole numbers?” Although it would not be a common alteration for discussing integer partitions in number theory, this “What-If-Not” modification fostered an opportunity to explore this problem from a new vantage point that led to deeper discoveries. Other educators (e.g., Whitin 2004) also have shared reports of mathematical opportunities provided by explorations that incorporate problem posing and students’ observations.

The group of students who previously used method 2 realized that their examples produced a general pattern. Because they had begun by choosing to divide 2012 into groups of equal size—say, 4—the product in consideration was $4^{2012/4}$; similarly, for other examples, they considered $2^{2012/2}$ or $6^{2012/6}$. Repeated examples helped students notice that they were exploring products of the form

$$2012 \div a = \frac{2012}{a} \rightarrow \left(a^{\frac{1}{a}}\right)^{2012}$$

Fig. 5 Algebra directs students to the best base.

$$\begin{aligned} 2^{1/2} &= 1.4142 \\ 3^{1/3} &= 1.44224 \\ 4^{1/4} &= 1.414 \\ 10^{1/10} &= 1.2589 \\ 2.5^{1/2.5} &= 1.44269 \\ 2.6^{1/2.6} &= 1.444126 \\ 2.9^{1/2.9} &= 1.44360 \\ 2.7^{1/2.7} &= 1.444655 \\ \rightarrow 2.8^{1/2.8} &= 1.4444 \\ 2.76^{1/2.76} &= 1.444632 \\ \textcircled{2.71}^{1/2.71} &= 1.444665 \\ 2.7111^{1/2.7111} &= 1.444665 \\ 2.719^{1/2.719} &= 1.444665 \\ 2.7109^{1/2.7109} &= 1.444665 \end{aligned}$$

Fig. 6 Iterating to approximate the maximum of $f(x) = x^{1/x}$ points to e.

$a^{2012/a}$. With some insight from algebraic manipulation (see **fig. 5**), the task of finding a maximum product was greatly reduced. Students needed to explore a single function.

Although the implications were not immediately obvious, students focused on the function inside the parentheses: $f(x) = x^{1/x}$. Groups reasoned that this function had to be as large as possible and that the constant exponent of 2012 would not affect the maximum partition. An exchange between students illustrates this point:

Student 1: Will putting in 2012 change [the maximum]?

Student 2: Well, no . . . we’re just looking to figure out what the x is. We would just do this part [i.e., $f(x) = x^{1/x}$], and then the 2012 will be maximized too.

Student 1: Oh, yeah, it won’t change [the answer].

At this point, students went down one of two paths, depending on their familiarity with calculus. Students less familiar with calculus recognized their task as essentially trying to find the largest value by guess and check, using calculations to get increasingly better approximations through an iterative process. For example, after reasoning that the maximum split should be between 2 and 3, one group started with $2.5^{1/2.5}$, which these students found to be larger than either $2^{1/2}$ or $3^{1/3}$. They continued, identifying that $2.6^{1/2.6}$ was even larger but that the result of $2.9^{1/2.9}$ went back down, providing a narrower range ($2.6 < x < 2.9$) for their search. This group continued in a semi-iterative fashion with increasing decimal accuracy, trying, among other possibilities, $2.75^{1/2.75}$ and $2.7111^{1/2.7111}$.

As is evident from **figure 6**, these students were able to justify that 2.71—and indeed, 2.718—would result in a larger product. As students’ interest waned in finding increasingly better estimates, the group was prompted to discuss the potentially infinite nature of the decimal expansion and if or when it might end. Students recognized limitations of the calculator for distinguishing better estimates (at some point, their efforts to maximize were indistinguishable on the calculator) but felt that the process for improving the estimates would continue indefinitely; a discussion comparing rational and irrational decimal expansions followed. Although the work from this approach cannot conclude with certainty that the best partition would be irrational, it did force students (in this case, primarily K–8 in-service teachers) to consider noninteger numbers as potential solutions to mathematical problems and reinforced differences between rational and irrational decimal expansions.

$$y = a^{\frac{1}{a}}$$

$$\ln y = \ln a^{\frac{1}{a}}$$

$$\ln y = \frac{1}{a} \ln a$$

$$\frac{1}{y} y' = \left(\frac{1}{a} \right) \left(\ln a - \frac{1}{a^2} \right)$$

$$y' = \frac{1}{a^2} + y \ln a \left(-\frac{1}{a^2} \right)$$

$$0 = \frac{1}{a^2} + a^{\frac{1}{a}} \ln a \left(-\frac{1}{a^2} \right)$$

$$\ln a = 1$$

$$a = e$$

Fig. 7 Logarithmic differentiation is used to locate the maximum.

Students more familiar with calculus, for whom the term “maximize a function” prompted solving for a value where the derivative function is zero, used a different path for evaluation. Although their initial attempts misused the power rule for polynomials, they eventually found the derivative of $f(x) = x^{1/x}$ by using logarithmic differentiation (see **fig. 7**) to obtain

$$f'(x) = x^{\frac{1}{x}} \cdot \frac{1}{x^2} (1 - \ln x),$$

which they set equal to 0 and solved:

$$f'(x) = x^{\frac{1}{x}} \cdot \frac{1}{x^2} (1 - \ln x) = 0$$

$$x^{\frac{1}{x}} \cdot (1 - \ln x) = 0$$

$$1 = \ln x$$

$$x = e$$

The result demonstrates that e gives the precise maximum of this function (one student shouted exuberantly, “Oh, $x = e$! It’s e ! Like 2.7, e !”), which echoes the approximations from the previous group and can be shown graphically as well (see **fig. 8**).

This outcome helps answer two current questions: What is the partition of a number into positive numbers (not necessarily positive whole numbers) that yields a maximum product? And, returning to the original problem, why is splitting into groups of 3 better than splitting into groups of 2? In particular, the best way to partition a number into positive numbers so that the product is a maximum is into groups of e . Theoretically, the maximum would be $e^{2012/e}$. Since $3^{1/3} > 2^{1/2}$, this analysis also helps explain why splitting into groups of 3 resulted in a larger product than splitting into groups of 2. However, because the irrational exponent in $e^{2012/e}$ does not represent a product in the way that e^3 represents $e \cdot e \cdot e$, the complete answer to the modified question about 2012 requires finding a rational approximation for e . Although time constraints limited any extended discourse on this topic, discussing the exact rational solution would be enlightening. Because $2012/e \approx 740.17$, there are two rational approximations of e to consider:

$2012/740 \approx 2.7189$ and $2012/741 \approx 2.7152$. The sum of 740 groups of 2012/740 and 741 groups of 2012/741 are both 2012; computing their products verifies that $(2012/740)^{740} > (2012/741)^{741}$.

The maximum product (requiring an integer exponent), then, is achieved at $(2012/740)^{740}$, where $2012/740 \approx 2.7189$ is the rational approximation of e that provides an exact solution. The theoretical maximum tends toward the irrational number e , although specific constraints may require a rational approximation. The precise solution to this problem could be used to promote further classroom discussion about irrational numbers and rational approximations for irrational numbers, a difficult concept to grasp (Arcavi, Bruckheimer, and Ben-Zvi 1987).

EXTENSION

Although this second “What-If-Not” problem has been discussed in relation to the ideas presented in method 2, the line of reasoning used in method 1 can also be generalized. In method 1, students reasoned about finding the maximum product partition with smaller numbers—for example, 12,

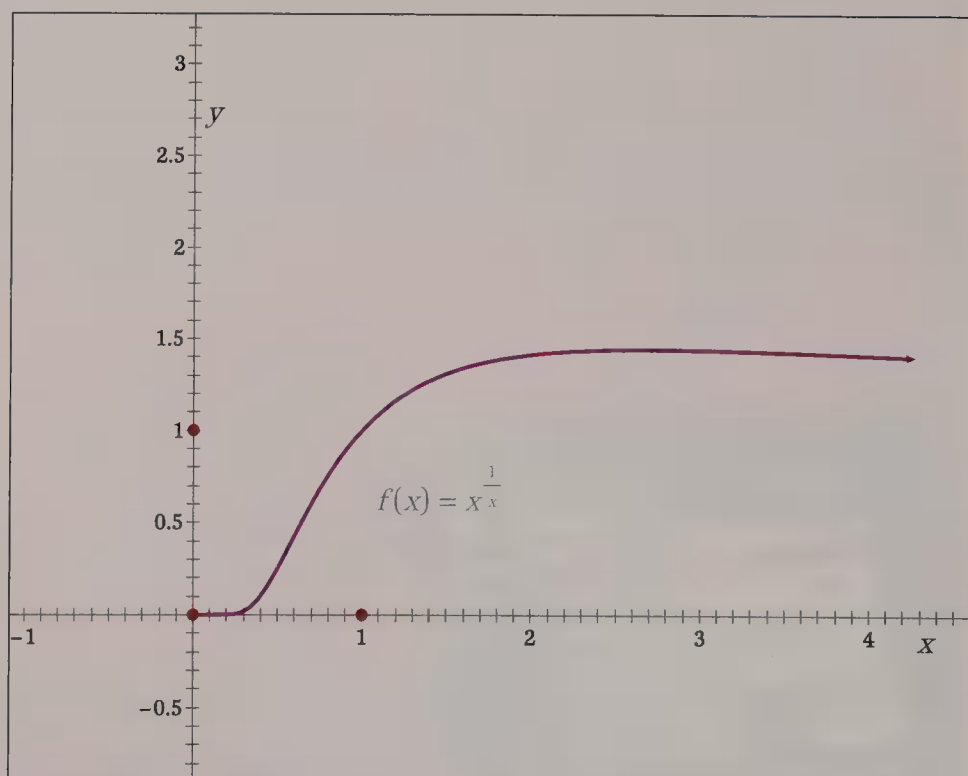


Fig. 8 Graphical representation shows that the maximum occurs at the irrational number e .

because $12 = 2 + 2 + 2 + 2 + 2 + 2 = 3 + 3 + 3 + 3$ but $2^6 < 3^4$. A composite number, expressed as the product of two factors, x and y , can be split not only into y x s but also into x y s. This observation is equivalent to recognizing that, for the given number,

$$\underbrace{x + x + x + \cdots}_y = \underbrace{y + y + y + \cdots}_x.$$

Because both sides sum to the same number, the question becomes, Which is bigger— x^y or y^x ? Assuming that $x^y < y^x$ (and knowing that both factors are positive) produces these equivalent statements:

- $\ln x^y < \ln y^x$
- $y \ln x < x \ln y$
- $\ln x/x < \ln y/y$

The two expressions in the last statement are of same form, and so the solution amounts to finding the maximum of another function: $f(x) = \ln x/x$. Maximization through graphical representations (see **fig. 9**) or calculus techniques ($f'(x) = (1 - \ln x)/x^2$) reiterates that e is the maximum product partition. (Indeed, the two functions, $f(x) = x^{1/x}$ and $f(x) = \ln x/x$, are meaningfully related.)

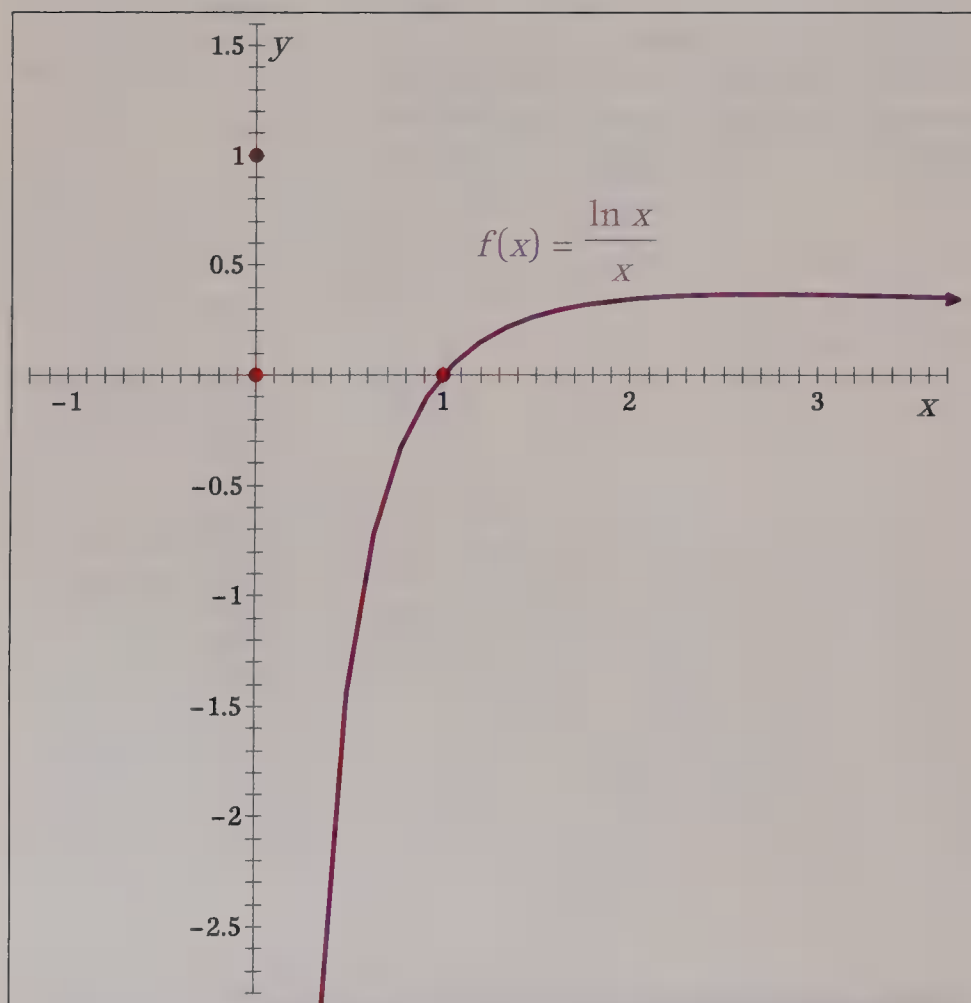


Fig. 9 A second graphical representation shows the maximum at $x = e$.

A DEEPER UNDERSTANDING

The original task—finding positive integers that sum to 2012 and whose product is as large as possible—was intended as an activity for in-service mathematics teachers. What started out as an exercise in problem solving turned into an opportunity to investigate the effect of problem posing; the consequences, notably, resulted in a much deeper discussion about numbers. Although partitions frequently refer to natural number partitions, letting go of that assumption and using the “What-If-Not” strategy for posing a modified problem (Brown and Walter 2005) fostered deeper insight into the original problem. The process and explorations forced students to refine the task (namely, that the task required finding the best base and not the largest exponent) and then to grapple with noninteger solutions. Consequently, students gained a deeper appreciation for the infinite nature of irrational numbers, generally, and properties of e , specifically. This activity typifies a “low entry, high ceiling” mathematical task, which can be a particularly effective means of engaging students in significant mathematical explorations.

Irrational numbers are the solution to many interesting problems in mathematics; connecting to students’ interest in the mathematics of finance is a practical and useful way to explore the irrational number e . The maximum product partition problem, which requires knowledge of only basic operations (i.e., sum, product, exponents), provides an alternative investigation that is accessible to younger students. The property of e encountered in this problem can be explored before the notion of limit is necessary. Although precise justification of e as the solution may be difficult with prealgebra or algebra students, calculators can be used to obtain better estimates of the maximum through crunching numbers or graphing to explore maximums of a function; both are manageable investigations for earlier ages. Indeed, the activity (perhaps with a number smaller than 2012) could motivate students to explore the growth of exponential functions, the meaning of fractional exponents, applications of logarithmic properties, or iterating between increasingly better rational approximations of irrational numbers. For calculus students, the problem provides a context for logarithmic differentiation and solving logarithmic equations.

In mathematics education, problems are often designed to have integer solutions. However, including and discussing interesting problems with solutions that are not integers may develop in mathematics students (and teachers) an appreciation for the infinite decimal expansions of irrational numbers and an understanding of their practical—and rational—place in a world of irrationals.

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Editor's note: For more on partitioning, see the MT Calendar problem for May 13, 2013, which appeared in the April 2013 issue of the journal.

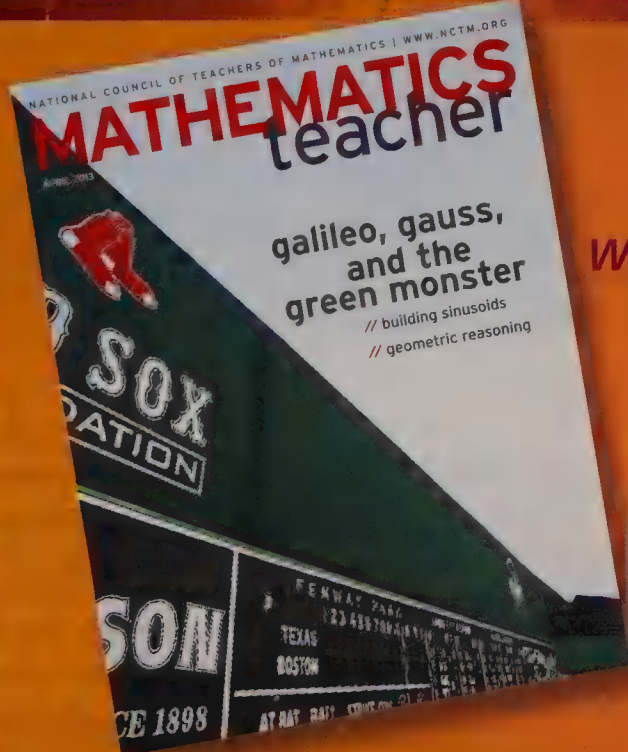


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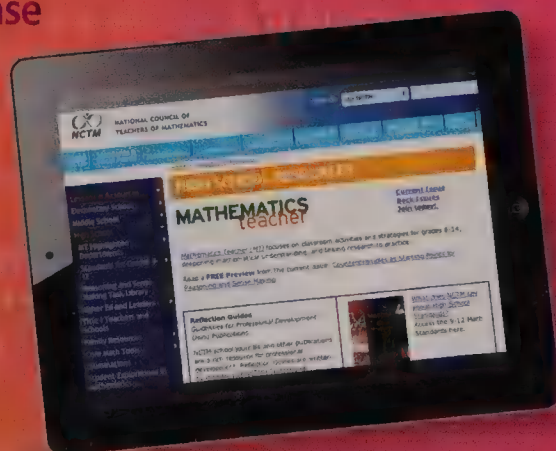


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Agida G. Manizade and

DEVELOP THE AREA TRAP

When calculating the area of a trapezoid, students use a range

A mathematics classroom that reflects the vision of NCTM's *Principles and Standards for School Mathematics* (2000) will have the teacher posing problems, asking questions that build on students' thinking, and encouraging students to explore different solutions. In teaching about area, it is not sufficient to give students the pertinent formulas and have them merely compute the areas of various polygons. The students should develop an understanding of the concept of area so that they can reason about the relationships between shapes to determine area.

According to the Common Core State Standards for Mathematics, students in sixth grade should be able to find the area of special quadrilaterals by composing them into rectangles or decomposing into triangles and other polygons. The Common Core Standards for Mathematics Practice also describe how students should construct viable arguments and critique the reasoning of others (CCSSI 2010).

One worthwhile task supporting these standards is to ask students to determine the area of a trapezoid and justify their method. We have used this ungraded exploration successfully with both middle school and high school students. Although students

arguerite M. Mason

OPPING EA OF A EZOID

of problem-solving strategies and measurement concepts.

do not often get a chance to discover the formula for the area of a trapezoid in their mathematics classrooms, we know that, when given an opportunity, they can come up with multiple ways of discovering or justifying this formula (LaSaracina and White 1999; Wanko 2005; Young 2010).

This activity can be used when the students are being introduced to the areas of special quadrilaterals, especially trapezoids. Before students engage with this task, we suggest that they explore and develop the formulas for the areas of triangles, rectangles, and parallelograms and gain some experience decomposing and recomposing polygons.

Alternatively, the activity may be approached as forging the connections between the various geometric and algebraic representations, making it a more advanced task suitable for high school students.

The class may be divided into small groups of two to three students. Groups are provided with poster paper and markers and challenged to produce at least one way of finding the area of a trapezoid and justifying their method. The teacher's main role is to observe each group, interjecting facilitating questions as appropriate. These questions could include the following:

1. In what ways could you decompose and recompose a trapezoid to create a shape or combination of shapes for which you already know the areas? What shapes would you use? Consider triangles, rectangles, and parallelograms.
2. Can you sketch a diagram that would help you describe the area of this trapezoid using the area formulas for a rectangle, a parallelogram, and a triangle?
3. Is there another way that you could decompose and recompose this trapezoid to create a shape or combination of shapes for which you already know the area or areas?
4. How does your method compare with other methods presented by your classmates? How is it similar? How is it different?

After time for the exploration, the groups can be asked to display their work and justify their ways of finding the area. Methods should be thoroughly compared and contrasted, until the class can show that the different formulas generated are equivalent.

One result of this activity is that each group develops an approach to justifying the formula for the area of a trapezoid that the students understand and can use to generate the formula later, if necessary. Students may prefer one method to another, depending on the level of their own geometry and algebra skills. The role of the teacher in this activity is as facilitator. The teacher may ask follow-up questions to clarify and extend the student's approach. The emphasis is on reasoning and sense making.

THE DEFINITION OF A TRAPEZOID

Two definitions of the term *trapezoid* are commonly used in the United States. One definition—a

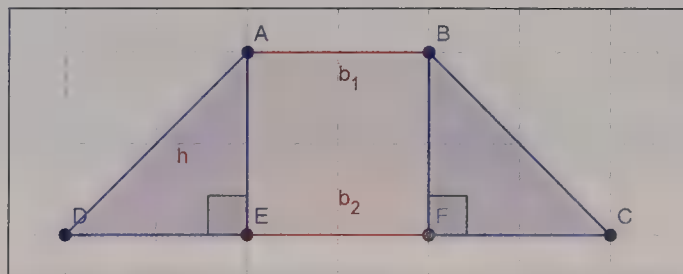


Fig. 1 This student solution shows a decomposition.

trapezoid is a quadrilateral with *at least* one pair of parallel sides—would include squares, rhombi, parallelograms, and rectangles. The other definition—a trapezoid is a quadrilateral with *exactly* one pair of parallel sides—excludes squares, rhombi, parallelograms, and rectangles from being classified as types of trapezoids. The latter definition is mandated by the Virginia Department of Education and is used in this activity, but either definition can be used with students' examples.

SPECIAL CASES

When presented with the task of developing a formula to calculate the area of a trapezoid, the students initially developed solutions that worked only for special cases, such as isosceles trapezoids and trapezoids with right angles.

One common strategy presented by students was to cut off two right triangles, leaving a central rectangle (see fig. 1). Extending that approach, some students cut a right triangle from one end of the trapezoid and moved it to the opposite end, creating a rectangle. It is important to recognize that this method works only for isosceles trapezoids (see fig. 2).

Other students began with a trapezoid containing a right angle and showed that it can be decomposed into a rectangle and a right triangle (see fig. 3). Then they added the areas of the rectangle and the triangle to derive the formula for the area of a trapezoid. This strategy works only for the special case of trapezoids with right angles.

As the students presented strategies that addressed only special cases, we followed up with questions to encourage all students to evaluate their strategies critically in the case of a general trapezoid. Doing so created an opportunity for all students to engage in mathematics reasoning at a deeper level and allowed us to facilitate a meaningful and rich mathematical discourse.

THE GENERAL CASE

Next, we challenged the students to develop a formula for computing the area of *any* trapezoid with base lengths b_1 and b_2 and height h (see fig. 4). Students' strategies can be categorized according to

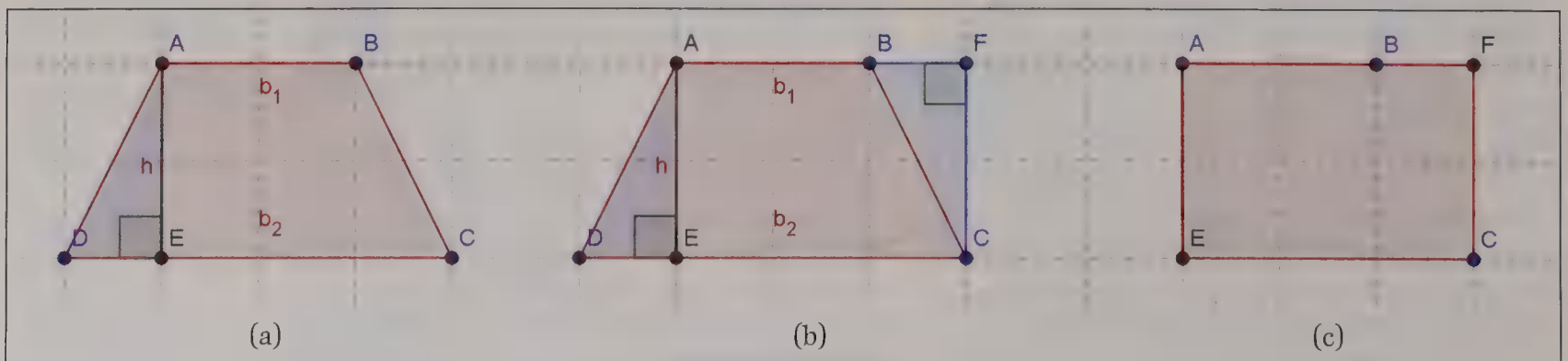


Fig. 2 Transforming the isosceles trapezoid into a rectangle dealt with a special case.

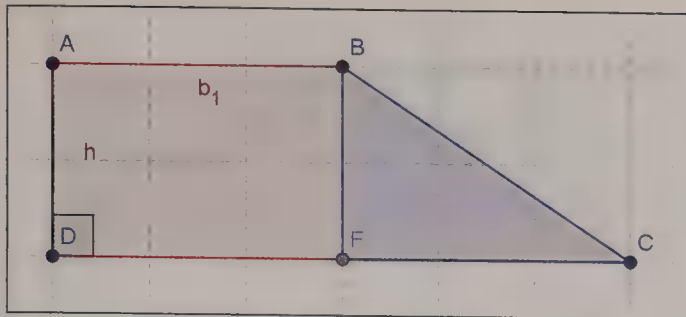


Fig. 3 If the trapezoid contains a right angle, this student solution is appropriate.

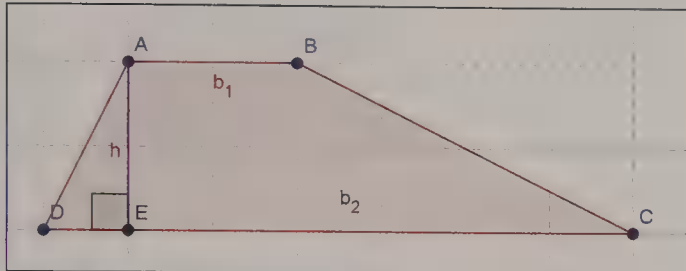


Fig. 4 A general trapezoid invites student analysis.

three types. A teacher should not expect to see as much variation in approach within a single class as is shown in **figures 5, 6, and 7**, but it is also unlikely that all the students will produce the same solution. In the event that they do create only one solution, the teacher can provide other examples and challenge the students to justify the appropriateness of these solutions, thus prompting the students to develop additional strategies beyond what they have been shown.

Type 1 Strategies: Decomposing into Simpler Shapes

In the type 1 proofs, students decomposed a trapezoid into simpler shapes and then found the area of the trapezoid as a sum (see **fig. 5**).

Some students decomposed the trapezoid into two triangles: $\triangle ACB$ and $\triangle ACD$ (see **fig. 5a**). If we combine the areas of these two triangles to make the trapezoid $ABCD$, the area is

$$\frac{1}{2}b_2h + \frac{1}{2}b_1h = \frac{1}{2}(b_1 + b_2)h.$$

Other students connected the midpoint I of the longer base CD to the vertices A and B , decomposing the trapezoid into three triangles (see **fig. 5b**). Since I is the midpoint of $\overline{CD}$, both DI and IC equal $b_2/2$. The area of trapezoid $ABCD$ may be found by combining the areas of the three triangles:

$$\begin{aligned} &\text{area } (\triangle ADI) + \text{area } (\triangle AIB) + \text{area } (\triangle BIC) \\ &= \frac{1}{2}\left(\frac{b_2}{2}\right)h + \frac{1}{2}b_1h + \frac{1}{2}\left(\frac{b_2}{2}\right)h \\ &= \frac{1}{2}(b_1 + b_2)h \end{aligned}$$

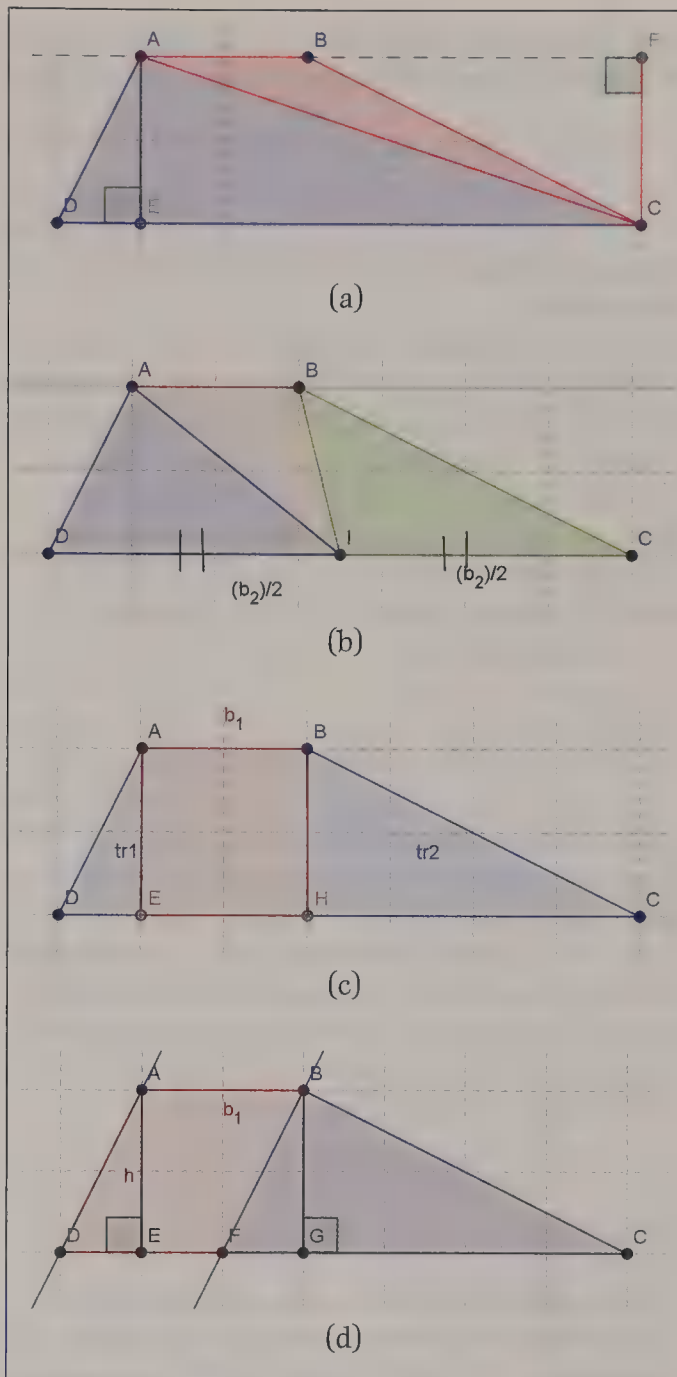


Fig. 5 Student strategies characterized as type 1 are shown here.

Still other students dropped perpendiculars from the vertices A and B to the opposite base, forming a rectangle and two right triangles (see **fig. 5c**). The two triangles can be slid together so that the altitudes AE and BH coincide. The base of the new triangle has length $b_2 - b_1$, and the area of the new triangle is $(1/2)(b_2 - b_1)h$. The sum of the areas of the rectangle and the new triangle gives the area of the original trapezoid:

$$\begin{aligned} b_1h + \left(\frac{1}{2}(b_2 - b_1)\right)h &= \left(b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_1\right)h \\ &= \frac{1}{2}(b_1 + b_2)h \end{aligned}$$

Some students decomposed the trapezoid into a parallelogram and a triangle by constructing a line parallel to leg AD through point B (see **fig. 5d**). The area of parallelogram $ABFD$ is b_1h . The area of

$\triangle BCF$ is $(1/2)(b_2 - b_1)h$. The sum of the areas gives the area of the original trapezoid:

$$b_1h + \frac{1}{2}(b_2 - b_1)h = \frac{1}{2}(b_1 + b_2)h$$

Type 2 Strategies: Using Transformational Geometry

In the type 2 strategies, students derived the formula for the area of the trapezoid by using transformational geometry. Four variations developed by students and summaries of their reasoning are shown in **figure 6**.

Some students transformed the original trapezoid into a triangle (see **fig. 6a**). Using the midpoint J of the side AD , they rotated $\triangle ABJ$ 180° about J , forming $\triangle BCI$ with the same area as the original trapezoid. The area of $\triangle BCI$ equals

$$\frac{1}{2}(ID + DC)h = \frac{1}{2}(b_1 + b_2)h.$$

Another student strategy was to duplicate the original trapezoid to create a parallelogram with twice the area (see **fig. 6b**). Trapezoid $ABCD$ is rotated 180° around the midpoint of side BC , forming parallelogram $AD'A'D$. The length of bases AD' and DA' is $b_1 + b_2$, so the area of parallelogram $AD'A'D$ is $(b_1 + b_2)h$. Since the area of the original trapezoid is half the area of parallelogram $AD'A'D$, it is

$$\frac{1}{2}(b_1 + b_2)h.$$

Other students constructed a different parallelogram that has the same area as the original trapezoid (see **fig. 6c**). Connecting the midpoints F and S of the legs of the original trapezoid forms the median (also known as the midsegment or midline). The smaller trapezoid $ABSF$ is rotated 180° around point S to form parallelogram $FE'A'D$, which has the same area as the original trapezoid. The area of parallelogram $FE'A'D$ is

$$(b_2 + b_1)\left(\frac{1}{2}h\right) = \frac{1}{2}(b_1 + b_2)h.$$

A fourth student strategy was to form a rectangle by constructing line segments perpendicular to the trapezoid base, through the midpoints F and G of each leg (see **fig. 6d**). The resulting triangles, $\triangle FID$ and $\triangle GCK$, were rotated 180° around F and G , respectively, to form the rectangle $HJKI$, which has the same area as the original trapezoid. The base length of the rectangle is the average of the two base lengths of the trapezoid, $(b_1 + b_2)/2$. Therefore, the area of the rectangle (and of the trapezoid) is

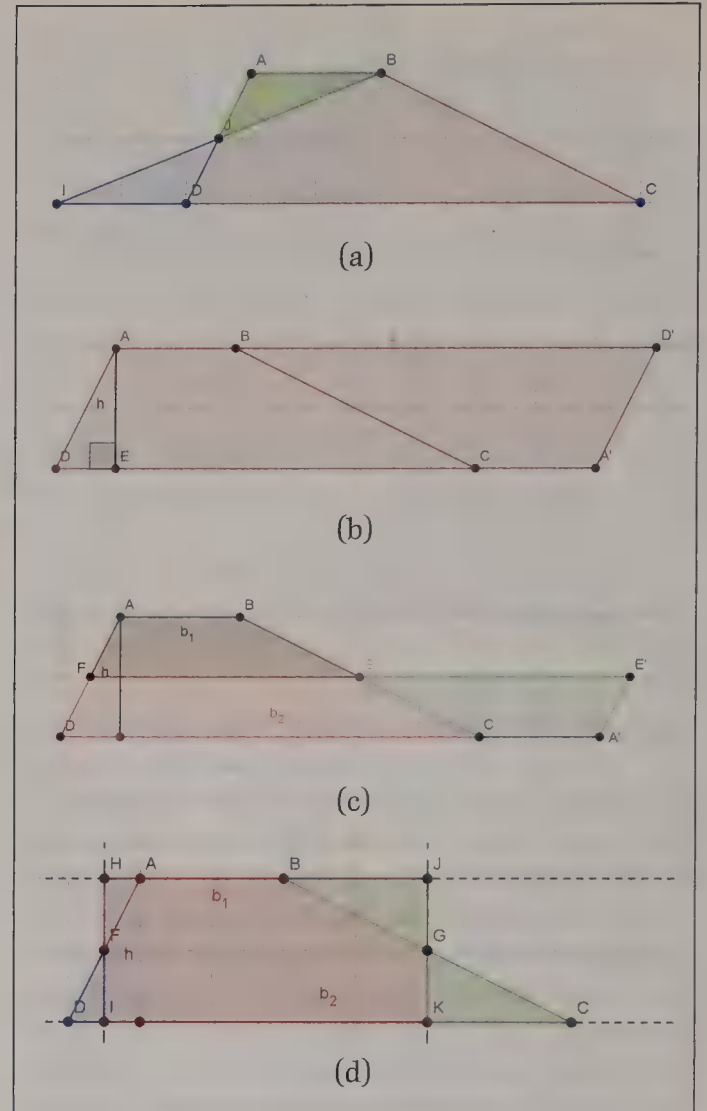


Fig. 6 Transformational geometry characterized type 2 approaches.

$$\frac{(b_1 + b_2)}{2}h = \frac{1}{2}(b_1 + b_2)h.$$

Type 3 Strategies: Enclosing the Trapezoid

Finally, in the type 3 strategy, students constructed a larger shape around the original trapezoid. Then students found the area of the trapezoid by subtracting the area of external pieces from that of the enclosing shape. We present three examples of solutions involving type 3 strategies below (see **fig. 7**).

Some students raised perpendiculars from endpoints of the longer base to form an enclosing rectangle (see **fig. 7a**). The area of the rectangle is b_2h . The two newly created triangles have total area $(1/2)(b_2 - b_1)h$. The area of the trapezoid can be calculated by subtraction:

$$b_2h - \frac{1}{2}(b_2 - b_1)h = \frac{1}{2}(b_1 + b_2)h$$

Another student strategy was to extend legs DA and CB until they met at point F , forming an enclosing triangle FDC (see **fig. 7b**). The newly added $\triangle FAB$ has altitude FG with length h_2 . The area of the small triangle $\triangle FAB$ is $b_1h_2/2$. The area

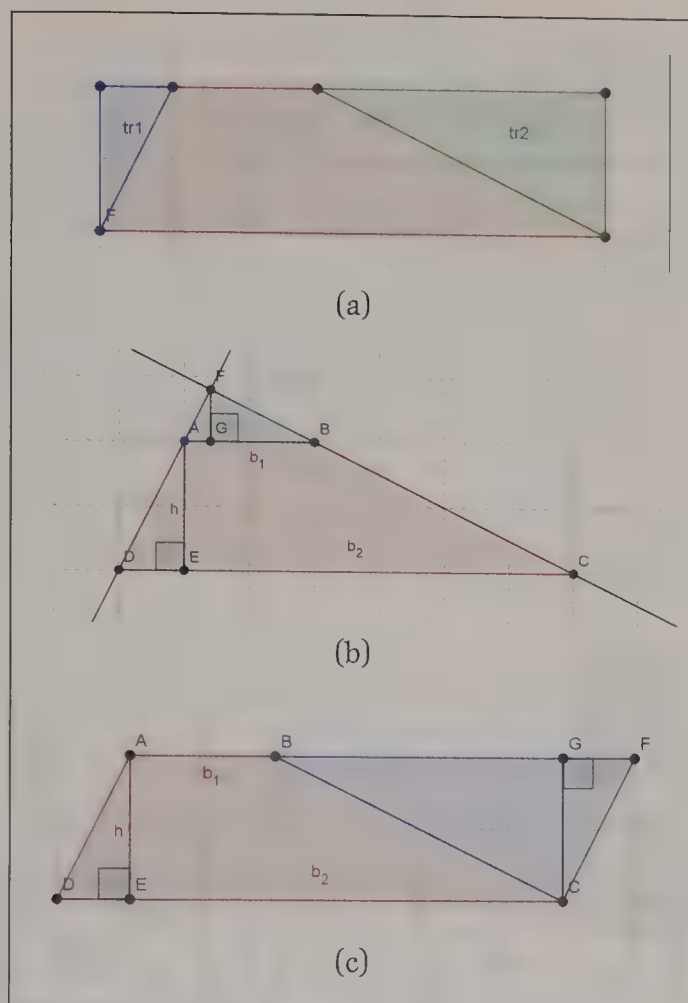


Fig. 7 Type 3 strategies involved building around the trapezoid.

of the large triangle $\triangle FDC$ is $b_2(h + h_2)/2$. The area of the original trapezoid is the difference

$$\begin{aligned} \frac{1}{2}b_2(h + h_2) - \frac{1}{2}b_1h_2 &= \frac{1}{2}b_2h + \frac{1}{2}b_2h_2 - \frac{1}{2}b_1h_2 \\ &= \frac{1}{2}b_2h + \frac{1}{2}(b_2 - b_1)h_2. \end{aligned}$$

To simplify this expression, we must use the similarity of the small and large triangles. Since $\overline{AB}$ is parallel to $\overline{CD}$, $\triangle FAB$ is similar to $\triangle FDC$. Therefore,

$$\frac{h_2}{b_1} = \frac{h_2 + h}{b_2}.$$

Solving for h_2 , we get

$$h_2 = \frac{b_1h}{b_2 - b_1}.$$

Substituting this value for h_2 into the expression for area of the trapezoid, we get

$$\begin{aligned} \frac{1}{2}b_2h + \frac{1}{2}(b_2 - b_1)\left(\frac{b_1h}{b_2 - b_1}\right) &= \frac{1}{2}b_2h + \frac{1}{2}b_1h \\ &= \frac{1}{2}(b_1 + b_2)h. \end{aligned}$$

A third approach was to enclose the original trapezoid in a parallelogram (see **fig. 7c**). Students constructed a line through point C parallel to leg AD . They extended the upper base b_1 to form parallelogram $AFCD$ with area b_2h . The area of the added triangle BFC is $(1/2)(b_2 - b_1)h$. The area of trapezoid $ABCD$ is the difference of the areas of parallelogram $AFCD$ and $\triangle BFC$, or

$$\begin{aligned} b_2h - \frac{1}{2}(b_2 - b_1)h &= b_2h - \frac{1}{2}b_2h + \frac{1}{2}b_1h \\ &= \frac{1}{2}(b_1 + b_2)h. \end{aligned}$$

This entire activity is flexible and can be used with students of varied mathematical backgrounds. A student's level of geometry development and algebra skills might elicit one type of strategy over another. Asking students to think about strategies presented by peers and to compare those with their own can generate a rich mathematical discussion. They have an opportunity to reason mathematically and communicate their ideas.

VAN HIELE LEVELS OF THINKING

This activity can be viewed in the context of reasoning and sense making as described in the Common Core Standards for Mathematical Practice. In this exploration, students who might first consider special cases of trapezoids that simplify the given task have the chance to gain insight into more generic examples.

The approach that an individual student uses when trying to determine the area of a trapezoid may depend on his or her van Hiele level of geometric thinking (Mason 1998). We surmise that students who are thinking at the visualization level (level 1), in which shapes are recognized by appearance alone, may decompose the trapezoid into familiar shapes, such as triangles and rectangles. Students who are thinking at the analysis level (level 2), in which properties of shapes are perceived but are isolated and unrelated, may try to use the fact that the altitude of the trapezoid is perpendicular to its bases or that the bases of the trapezoid are parallel. Students who are thinking at the abstraction level (level 3), in which definitions are meaningful and relationships are perceived between figures and between properties, may be more capable of creating approaches that involve transformations and addition of figures (type 2 and type 3). The variations in approach are to be expected and even encouraged. What is most important is that students are able to justify their thinking.

Some students might prefer to manipulate objects physically to see the connections between the area of the trapezoid and its algebraic formula.

Mathematically proficient students can determine the correspondence between geometric manipulations and algebraic equations; they can develop more than one approach to producing the area of a trapezoid and will be able to see correspondences between different methods.

Formulating the area of a trapezoid is a worthwhile task that supports further understanding of geometry, modeling, and proof, especially reasoning and sense making. It requires active participation of all students and allows teachers to act as facilitators who can create a rich mathematical discourse in the classroom. The model of this activity could also be used to design other activities that promote reasoning and sense making.

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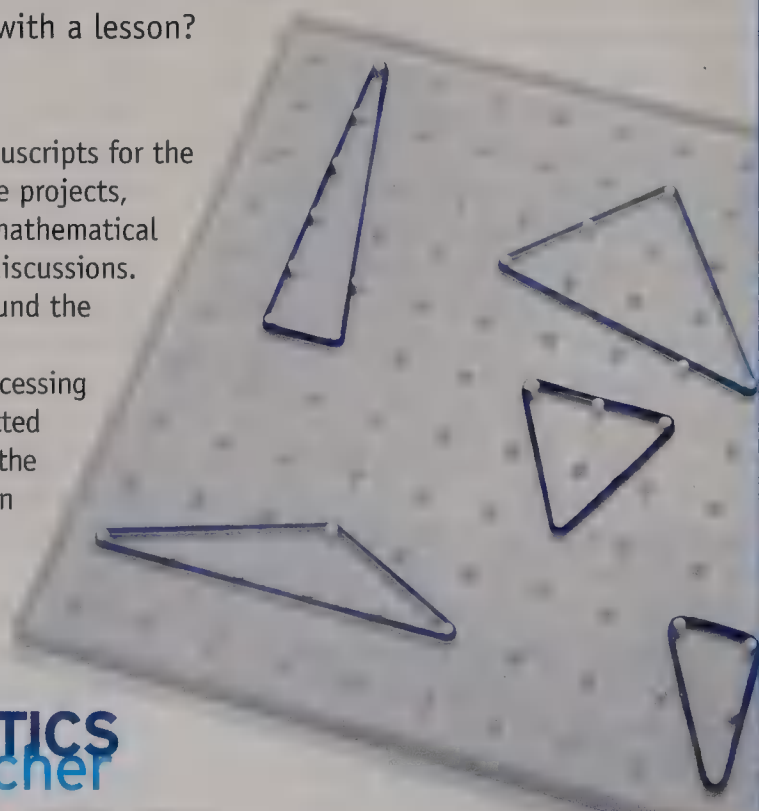
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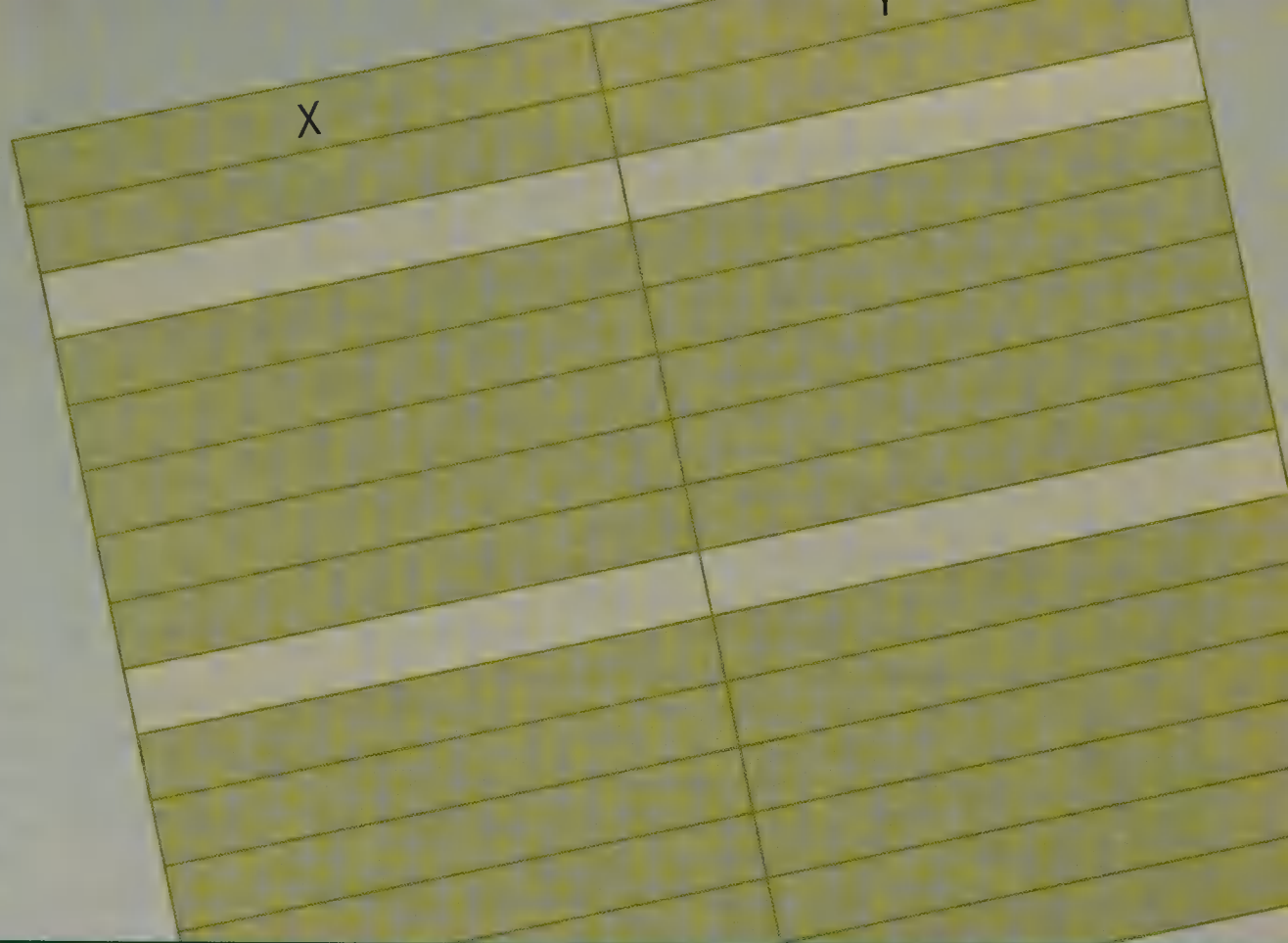


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Using Covariation Reasoning to Support Mathematical Modeling

Table representations of functions allow students to compare rows as well as values in the same row.

Erik Jacobson

For many students, making connections between mathematical ideas and the real world is one of the most intriguing and rewarding aspects of the study of mathematics. In the Common Core State Standards for Mathematics (CCSSI 2010), mathematical modeling is highlighted as a mathematical practice standard for all grades. To engage in mathematical modeling, beginning algebra students must learn to use their understanding of arithmetic operations to make mathematical sense of problem situations and to relate this sense making to functions represented by equations, tables, and graphs. The word problems commonly used in beginning algebra courses give opportunities to practice mathematical modeling. Further, the ability to reason with quantities as well as numbers is an important capacity for students to develop.

Quantities are conceptions of things that can be measured, such as distance or time. The measure of a quantity has a defined unit and a process for assigning a number that represents the proportional relationship between a particular value of the quantity and the unit (Thompson 2011). Quantitative reasoning is different from numerical reasoning because quantitative reasoning involves a clear mental image of how quantities are related (Thompson 2011). Someone who quantitatively understands the average speed of a sprinter who runs 100 meters in 4 seconds might imagine the 100-meter track divided into 4 sections, each 25 meters long, and then imagine the runner traversing one section during each second. By contrast, if students compute $100 \text{ m}/4 \text{ s} = 25 \text{ m/s}$, they may understand only the arithmetic relationship between the numbers 100, 4, and 25 without

understanding why dividing 100 by 4 makes sense in this situation. Quantitative reasoning is a key resource for students who are learning to use algebra to model relationships between quantities that vary.

Two kinds of quantitative reasoning have a special relevance for beginning algebra students. The *correspondence* perspective deals with the question, How is one quantity related to another? A correspondence understanding of speed might be expressed as the rule that relates each value for time with a unique value for distance, such as the equation $y = 25x$, where x represents time and y represents distance. By contrast, the key question for *covariation* reasoning is, How does one quantity change as another quantity changes? A covariation understanding of speed would focus on how distance and time change together—that is, the distance covered increases by 25 meters as the elapsed time increases by 1 second.

Both kinds of reasoning are important goals for algebra students. Correspondence is a fundamental piece of mature reasoning about functions, and covariation is critical for developing the rate-of-change concept. Research shows that covariation is a common entry point into algebra for students (e.g., Confrey and Smith 1995); but traditional approaches to teaching algebra emphasize correspondence and often have little or no treatment of covariation (Smith 2003). Thus, this article focuses on students' use of covariation and how to support it in the classroom.

Presented here are two sessions from Ms. Holmes's classroom (the teacher's name is a pseudonym) in which seventh graders intuitively used covariation to begin to make sense of word problems. These passages show how students' covariation reasoning might surface in the classroom and illustrate some of the teaching strategies that Ms. Holmes used to support her students' reasoning. The sessions also provide a foundation for the discussion of classroom strategies, which summarizes research-based strategies for supporting students' use of covariation reasoning to build robust mathematical models.

CLASSROOM SESSION 1

In session 1, a small group of students was working on problem 1 during a lesson spent reviewing for a state achievement test.

The teacher's sequence of questions implied a comparison between the paired quantities. The student's responses indicate a comparison between rows in the hypothetical table of values.

Problem 1: Sally needs $3\frac{2}{7}$ yards of fringe to trim each drape. If she has 8 drapes, how much fringe does she need? What operation is used to solve this problem?

- (a) addition
- (b) subtraction
- (c) multiplication
- (d) division

The first student started by guessing that the operation was subtraction but after some time changed his mind.

Student 1: I don't think you subtract now.

Ms. Holmes: What do you [other] guys think?

Student 2: Um . . . divide?

Student 3: Yeah.

Ms. Holmes: Divide? Why?

Student 2: Because . . . [thirty-five seconds elapse] . . . I don't know.

Rather than challenging student 1, Ms. Holmes asked for other students' ideas. She also pressed students to justify their answer of division.

To help the students make progress, Ms. Holmes next read the problem out loud, asked the students to explain the problem in their own words, and had them draw a picture of the situation. After a few minutes, the students were still stuck, so she asked a sequence of questions:

Ms. Holmes: So, if Sally had 1 drape, how many yards would she need?

Student 1: Three and two-sevenths [$3\frac{2}{7}$] . . .

Ms. Holmes: All right. What if she had 2?

Student 1: She would need . . . twice that $3\frac{2}{7}$.

Ms. Holmes: [nods] What if she had 3?

Student 1: Twice . . . I mean, 3 times that.

Ms. Holmes: So, what are you doing each time?

Student 1: Multiplying . . . Oh! . . . So you multiply. It's multiplication.

Ms. Holmes then asked the two other students to explain why multiplication was the appropriate operation.

Discussion of Session 1

Each of Ms. Holmes's questions asked student 1 to compare a quantity of drapes with the corresponding quantity of fringe, so each of these questions is about correspondence. Neither Ms. Holmes nor the

students used a table of values for this problem, but it is natural to imagine a hypothetical table (see **fig. 1**). In such a table, each of Ms. Holmes's questions asks about values within a single row.

What is interesting about this episode is that the sequence of questions implied a comparison between the paired quantities. The implicit reasoning supporting student 1's statement that 2 drapes would need "twice that $3\frac{2}{7}$ s" is that 2 drapes are twice as many as 1 drape. The student's responses indicate that he recognized that as the quantity of drapes doubles and triples, the quantity of fringe is multiplied by 2 and then by 3. This reasoning can be understood as a comparison between rows in the hypothetical table of values (see **fig. 1**).

The final part of the episode involved looking back over the sequence of examples. Ms. Holmes's final question was probably intended as a general correspondence question, with "each time" referring to each correspondence between the number of drapes and the amount of fringe. The student likely interpreted "each time" to refer to each new pair of drapes and fringe and compared the new pair with the initial one. In any event, by asking a sequence of specific questions and then asking the student to reflect across these examples, Ms. Holmes was able to build this student's covariation reasoning and help him establish a meaningful mathematical model of the quantities in problem 1.

CLASSROOM SESSION 2

The class discussion of problem 2 occurred on the same day but during a different period and with different students. Earlier in the year, these students had worked on distinguishing directly and inversely proportional relationships and on writing linear equations using tables of corresponding x - and y -values.

Problem 2: Juan can clean up after the party in 2 hours if he works alone, but he hopes his friends will help. Write an equation relating the number of people (x) and the amount of time (y) it would take to clean up if everyone works at the same rate.

One student initially guessed that the equation was $y = 2x$. However, Ms. Holmes pointed out that this equation would not work because as the number of people increased, the amount of time should decrease. The students were still stuck, so to direct the students' reasoning about the problem, Ms. Holmes constructed a table on the whiteboard that included three values for x (the number of people): 1, 2, and 4 (see **fig. 2**). As she questioned the students, they readily agreed that one person would take 2 hours and that 2 people would take just

Correspondence (Ms. Holmes)	
So if she had one drape, how many yards would she need?	
What if she had two?	
x (number of drapes)	y (yards of fringe)
1	$3\frac{2}{7}$
2	$6\frac{4}{7}$
3	$9\frac{6}{7}$
8	?

Covariation (Student 1)
 She would need twice that $3\frac{2}{7}$.

Fig. 1 The teacher intends to build a correspondence between quantities in the same row. The student sees a covariation between rows.

Correspondence (Ms. Holmes)	
So what are we doing here each time?	
x (number of people)	y (number of hours)
1	2
2	1
4	?

Covariation (Student 4)
 Each time it goes down, it goes down by, like, halves.

Fig. 2 The student generalizes (incorrectly) using covariation.

1 hour, but they disagreed about the time it would take 4 people.

Ms. Holmes: What if you had 4 people?

Student 1: Thirty [30] minutes.

Ms. Holmes: How are you getting that?

Student 1: It's . . . um . . .

Student 2: It'd be 15 minutes . . . because it's half of . . . because if you have 3 people, it would be 30, and 4 it would be 15.

Ms. Holmes: Explain to me what you're thinking.

Student 3: x divided by two equals y .

Ms. Holmes: x divided by 2? [points to the first row of the table in **fig. 2**] Well, 1 divided by 2 is $\frac{1}{2}$. . .

Student 3: Oh, it's y divided by 2 equals x !

Ms. Holmes: Well, 2 divided by 1 is 2, but 2 divided by 2 . . .

Student 4: [interrupting] Each time it goes down, it goes down by, like, halves.

Ms. Holmes: So, what are we doing here each time [pointing to x and then y in the table]?

Student 5: Going down by a half each time.

Only a few minutes were left in the period, so Ms. Holmes went on to remind the class of the equation $y = k/x$ (the correspondence rule). She

showed that this rule was algebraically equivalent to $yx = k$. Finally, she demonstrated that the product of x and y was always 2 in this problem.

Discussion of Session 2

In this episode, Ms. Holmes pressed students to explain their thinking and offered counterexamples when students guessed incorrectly. In this way, she enabled students to describe their thinking and also promoted mathematical reasoning.

Just as in classroom session 1, the key was Ms. Holmes's sequence of questions about correspondence. As before, several students were reasoning about covariation in the context of the problem. Student 2 hypothesized that the value for 3 people was $1/2$ hour, or 30 minutes, and his hypothesized time for 4 people was half the previous value— $1/4$ hour, or 15 minutes—perhaps because he believed that each new person would cut the time in half. Student 4 apparently picked up on this idea and made a more general covariance statement.

The final part of the episode involved looking back over the sequence of examples. Ms. Holmes's final question was intended as a general correspondence question, just like her final question of the first episode. Her gestures indicated that "each time" referred to each row relating a specific number of people and the corresponding amount of time. However, student 5 reasoned about covariance instead of correspondence and interpreted "each time" to mean each new row. He compared each new row with the previous one, claiming that time was "going down by a half each time." Unfortunately, the period ended before these students had an opportunity to differentiate correspondence and covariation reasoning about the problem situation or find the correct mathematical model.

STRATEGIES FOR THE CLASSROOM

As we review the classroom episodes and the research literature (e.g., Carlson and Oehrtman 2005; Ellis 2011), several strategies emerge that teachers can use to support students' covariation reasoning:

1. Use a sequence of specific pairs of values to support students' reasoning about the problem.

Developing the ability to clearly and explicitly reason about quantities that are changing together will support students in beginning algebra and lay the foundation for later success in mathematics.

2. Ask students to describe and explain their thinking about a single pair of values and to compare different pairs of values. Listen carefully for covariation or correspondence reasoning.
3. If students make incorrect claims, ask for other students' ideas. Model quantitative reasoning by providing a mathematical reason or counterexample based in the problem situation that explains why the claim is incorrect.
4. Support covariation reasoning by asking students the following kinds of questions about problem situations (adapted from Carlson and Oehrtman 2005):

- What quantities are changing together, and how are they changing?
- As one quantity increases, does the other quantity increase or decrease?
- As one quantity increases in constant increments, by what amount does the other quantity change?
- As one quantity increases in constant increments, what is the rate of change in the other quantity?

Selecting specific examples (strategy 1) so that one quantity is changing in constant increments might emphasize covariation relationships for students, such as Ms. Holmes's choice of 1, 2, and 3 for problem 1. On the other hand, the choice of a doubling sequence 1, 2, and 4 for problem 2 may have contributed to the students' misunderstanding of the relationship between people and time.

As teachers listen carefully and help students communicate clearly about covariation, there will be opportunities for mathematical exploration that can enrich students' conceptions of functions. For example, students 2, 4, and 5 in episode 2 were actually describing a different function that could be represented with the correspondence rule, $y = 2(1/2)^{x-1}$, or in terms of covariation (y decreases by a factor of $1/2$ as x increases by 1). A teacher might ask students in the class to compare this function with the correct function for problem 2: $y = 2/x$.

REASONING ABOUT COVARIATION IS CRUCIAL

Reasoning about covariation is not a crutch; it is a crucial skill. Developing the ability to clearly and

explicitly reason about quantities that are changing together will support students in beginning algebra and lay the foundation for later success in mathematics. Students who intuitively use covariation to think about how quantities change may not have access to simple, unambiguous language to describe their thinking. When using tables to develop students' understanding of functions, teachers can help students describe, explain, compare, and relate covariation reasoning between rows and correspondence reasoning within rows.

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Connecting Research to Teaching

Bridging the Divide

Have you ever noticed a gap between research and practice?

How can research effect change in the classroom?

The Connecting Research to Teaching department of *MT* invites classroom teachers to explore research findings in relation to their practice. We also invite education researchers to demonstrate how results from their studies shape classroom practice. Findings from collaborative action research projects are also encouraged. Evidence of connections from research to practice commonly includes student work and brief transcripts from interviews or classroom videos.

Some ideas to consider:

- Does your research have the potential to change classroom practice?
- What urgent questions emerge in learning environments?
- In what ways is current research reflected in the classroom?
- How does research inspire innovation in the classroom?

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Manuscripts should include the research context and should be written with the classroom teacher as reader in mind. A good way to focus your material is to identify one clear message for teachers. Sharing student work where appropriate is encouraged. Manuscripts must be 2500 words or fewer.

See <http://www.nctm.org/news/content.aspx?id=22602> for submission guidelines. If you have questions about whether a particular idea would be a good fit for the department, contact Margaret Kinzel, mkinzel@boisestate.edu, or Laurie Cavey, lauriecavey@boisestate.edu.



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CALL FOR MANUSCRIPTS

How many fewer digits will the base-16 representation of 2^{100} have compared with the base-2 representation of the same number?

1

Determine without a calculator which, if any, of the following integers are divisible by 44:

838, 4004, 4040, 8580, 22944, 54296

5

Confirm that the expression

$$\sqrt{2 - \sqrt{3}}$$

is equal to the expression

$$\frac{\sqrt{6} - \sqrt{2}}{2}.$$

9

The expansion of $(a + b + c)^n$, where n is a positive integer, has 406 terms. Find the value of n .

Find the locus of points equidistant from the point $(4, 2)$ and the point $(7, 0)$.

2

Find the locus of points equidistant from the point $A(4, 2)$ and the line $x = 7$.

6

Find the locus of points twice as far from the point $A(4, 2)$ as from the line $x = 7$.

10

What is the surface area of the largest sphere that fits inside the largest cylinder that fits inside a cube with surface area 8?

How many *lattice points* (points with integral coordinates) lie on or in the interior of the triangle with vertices $(0, 0)$, $(0, 6)$, and $(6, 0)$?

3

Find the perimeter of the *dart* (or nonconvex kite) formed by the intersection of the following graphs:

$$f(x) = \frac{1}{2}|x + 6| - 4$$

and

$$g(x) = 2|x + 6| - 10.$$

7

Right triangle ABC has $m\angle A = 30^\circ$, $m\angle B = 60^\circ$, and the length of hypotenuse $AB = 1$. Extend side AC through C to a point D so that $AD = 1$. Find an expression for the length of $\overline{BD}$ without using trigonometry. Use the result of the problem of March 9 to simplify your expression.

11

The points $(4, 2)$, $(-1, -3)$, and $(-10, 6)$ are the midpoints of the sides of $\triangle ABC$. What is the area of $\triangle ABC$?

Write an expression, in terms of k , for the number of lattice points that lie on or in the interior of the triangle with vertices $(0, 0)$, $(0, k)$, and $(k, 0)$.

4

A dart is formed by the intersection of the graphs

$$f(x) = \frac{1}{2}|x + 6| - 4$$

and

$$g(x) = k|x + 6| - 10.$$

If the area of the dart is 18, what is the value of k ?

8

Consider that the expression

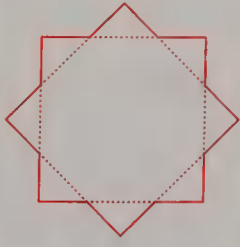
$$(a + b + c)^{10}$$

can be rewritten as

$$((a + b) + c)^{10}.$$

How many terms are there when $(a + b + c)^{10}$ has been expanded and simplified?

When a square is rotated 45° about its center, a nonconvex 16-gon is formed. If the original square has side length 16 in., what is the perimeter of the 16-gon?



16

How many terms are there in the expansion of $(a + b + c + d)^{42}$?

$17a$ is a perfect square, where a is a positive integer. What is the smallest value of a such that $17 + a$ is also a perfect square?

17

Find the locus of points twice as far from the line $x = 7$ as from the point $A(4, 2)$.

Given: x is any real number such that the expressions below are defined on the real numbers. Which of these expressions will always have a value greater than 0?

$$|x + 4| - |x|, |x + 2| - |x - 2|, x^2, \frac{1}{\sqrt{x}}$$

24

The sum of the proper divisors of a *perfect number* is equal to the number itself. For example, $6 = 1 + 2 + 3$, and $28 = 1 + 2 + 4 + 7 + 14$. There are only two additional perfect numbers less than 1,000,000. Solve the system to find these two perfect numbers; then confirm that they are perfect:

$$16a + 28 = b - 164$$

$$\frac{b}{127} = \frac{a - 112}{6}$$

28

The dartboard shown was formed by connecting midpoints. For example, the vertices of $\triangle DEF$ and $\triangle JKL$ are the midpoints of the sides of $\triangle ABC$ and $\triangle ADF$, respectively. If a player without skill throws a dart that hits the board, what is the probability that the dart lands in a shaded region?



25

$\triangle ABC$ has incenter P . The angle bisectors and three radii perpendicular to the sides of $\triangle ABC$ form six triangular puzzle pieces. Construct a figure like the one below and cut out the six puzzle pieces. Rearrange them to form one

rectangle and write an expression for its area.



29

Jack claims that if he tosses three fair coins, he can get exactly two heads. Jill bets that she could do something even harder—get exactly two heads if she tosses four fair coins. Jack says that his task is harder. Who is right? Does your answer change if the coin is not fair and has a .6 probability of landing heads?

18

Find the smallest positive integer x such that $x + 2014$ is a palindrome. Find the smallest positive palindrome that results in another palindrome when added to 2014.

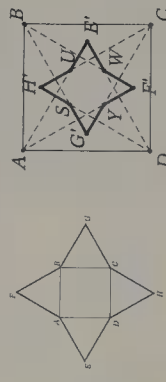
How many distinguishable arrangements of the letters in the word

MATHEMATICS

are possible?

26

Square $ABCD$ has an equilateral triangle on each of its sides, forming a star shape, $AFBGCHDE$. Construct equilateral triangles internally to get the star $YF'WE'UH'SG'$, which is similar to $AFBGCHDE$. Find the ratio of similitude.



30

How many ways can 4 pieces of candy be distributed among Ahmes, Bhaskara, Cardano, and Diophantus if none of the pieces may be cut? Giving all 4 pieces to Ahmes is not the same distribution as giving all 4 pieces to Diophantus.

19

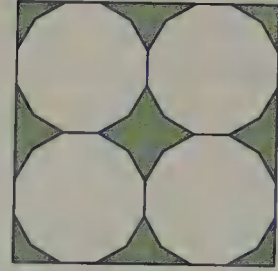
You roll 2 fair dice, one yellow and one blue. Let A be the event that the yellow die shows an even number. Let B be the event that the sum of the faces is less than 6. Are A and B independent events?

Robin has \$6.20 in nickels and quarters. If the nickels were quarters and the quarters were nickels, he would have \$3.20 more. How many quarters does Robin have?

23

A square patio, 24 ft. on a side, contains four concrete slabs in the shape of regular dodecagons, as shown in the figure. Any area not covered by

concrete contains shrub and flower plantings. Find the area containing plants.



31

Looking for more Calendar problems?

Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher*.

Edited by **Margaret Coffey**, Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and **Art Kalish**, artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Problems 12 and 13 were submitted by Richard Forringer (retired) of Durham Academy in Durham, North Carolina.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, \$22.95) and the 100 Problem Poster (stock number 13207, \$9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

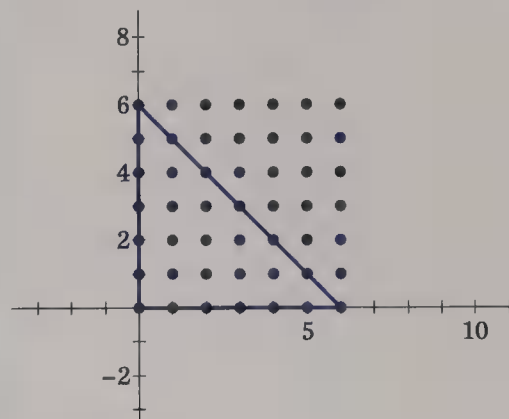
1. 75. The number 2^{100} written in base 2 is the digit 1 followed by 100 zeros, 101 digits in all. Since $2^{100} = (2^4)^{25} = 16^{25}$, the base-16 representation is the digit 1 followed by 25 zeros. So the base-16 representation has $101 - 26 = 75$ fewer digits.

2. $y - 1 = \frac{3}{2}\left(x - \frac{11}{2}\right)$. The set of points

equidistant from $A(4, 2)$ and $B(7, 0)$ forms the perpendicular bisector of segment AB . Since the slope of $\overline{AB}$ is $-2/3$, the perpendicular bisector has slope $3/2$ and passes through the midpoint of $\overline{AB}$, which is $(11/2, 1)$. One equation is

$$y - 1 = \frac{3}{2}\left(x - \frac{11}{2}\right).$$

3. 28. There must be $7 \cdot 7 = 49$ lattice points lying on or in the interior of the square with vertices $(0, 0)$, $(0, 6)$, $(6, 6)$, and $(6, 0)$. Since the slope of the segment connecting $(0, 6)$ to $(6, 0)$ is -1 , seven of those points lie on the diagonal. Thus, $(49 - 7)/2 = 21$ lattice points lie on either side of that diagonal. Therefore, $21 + 7 = 28$ points lie on or in the interior of the triangle with vertices $(0, 0)$, $(0, 6)$, and $(6, 0)$.



Alternate solution: The slope of the segment connecting $(0, 6)$ to $(6, 0)$ is -1 , so the hypotenuse of the right triangle must contain seven lattice points: $(0, 6)$, $(1, 5)$, $(2, 4)$, $\dots$, $(6, 0)$. On or below the hypotenuse are 7 lattice points with x -coordinate of 0; 6 lattice points with x -coordinate of 1; and so on. The total number of points is the sum of the integers from 1 to 7—that is, $8 \cdot 7/2 = 28$.

4. $(k + 2)(k + 1)/2$. The reasoning used to solve the March 3 problem can be used again here. There must be $(k + 1)^2$ lattice points lying on or in the interior of the square with vertices $(0, 0)$, $(0, k)$, (k, k) , and $(k, 0)$. Of those $(k + 1)^2$ points, $k + 1$ must lie on the segment connecting $(0, k)$ to $(k, 0)$. Thus,

$$\frac{k^2 + 2k + 1 - (k + 1)}{2} = \frac{k^2 + k}{2} \text{ lattice points}$$

lie on either side of that segment. Thus,

$$\begin{aligned} \frac{k^2 + k}{2} + k + 1 &= \frac{k^2 + 3k + 2}{2}, \\ \text{or } \frac{(k + 2)(k + 1)}{2}, &\text{ points} \end{aligned}$$

lie on or in the interior of the triangle with vertices $(0, 0)$, $(0, k)$, and $(k, 0)$. Observe that this expression is equal to the sum of the integers from 1 to $k + 1$.

5. 4004, 8580, 54296. If a number is divisible by 44, it must be divisible by both 4 and 11. Divisibility by 4 requires that the number formed by the final two digits is divisible by 4. Divisibility by 11 requires that the difference between the sum of the digits in odd positions and

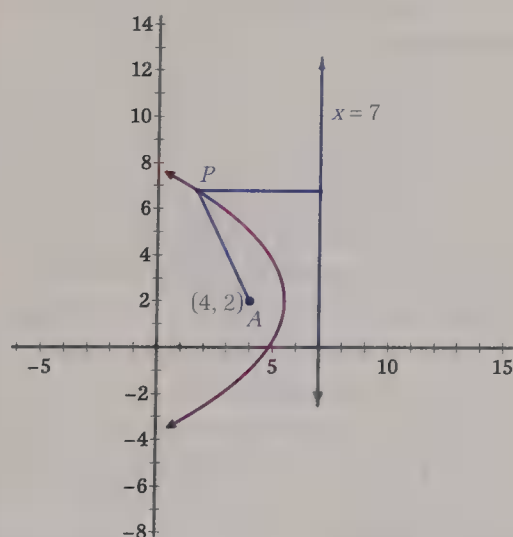
Solution 5

Integer	Sum of Odd-Numbered Digits	Sum of Even-Numbered Digits	Difference	Divisible by 11?
4004	4	4	0	Yes
4040	8	0	8	No
8580	16	5	11	Yes
22944	15	6	9	No
54296	13	13	0	Yes

the sum of the digits in even positions be divisible by 11.

Since 38 is not divisible by 4, 838 is not divisible by 44. The remaining five integers are divisible by 4, so we consider whether they are divisible by 11. The table for solution 5 organizes our test results.

6. $x = (-1/6)(y - 2)^2 + 11/2$. The locus is a parabola opening to the left, with focus (4, 2) and directrix $x = 7$. Since the distance from (4, 2) to the directrix is 3 units, the distance from focus to vertex is 1.5 units, and the vertex will lie at (11/2, 2). We have $(x - 11/2) = (-1/(4 \cdot 1.5))(y - 2)^2 \rightarrow x = (-1/6) \cdot (y - 2)^2 + 11/2$.



Alternate solution: Use the distance formula. The distance from a point (x, y) to the point (4, 2) is

$$\sqrt{(x - 4)^2 + (y - 2)^2},$$

and the distance from (x, y) to the line $x = 7$ is $|x - 7|$. Set the two expressions equal and square both sides: $x^2 - 8x + 16 + (y - 2)^2 = x^2 - 14x + 49 \rightarrow -6x + 33 = (y - 2)^2 \rightarrow x = (-1/6)(y - 2)^2 + 11/2$.

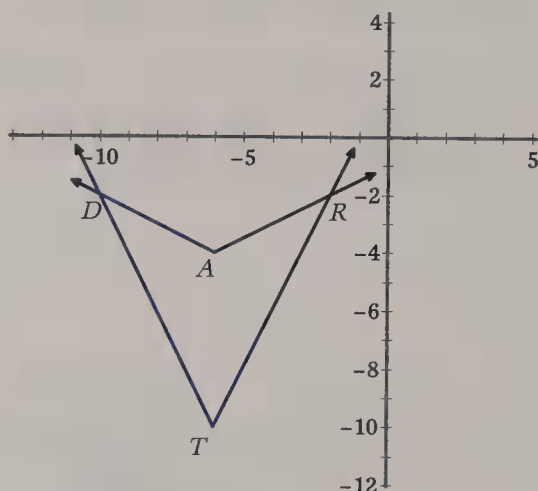
7. $12\sqrt{5}$, or about 26.83. The graphs of the functions indicate that the dart's vertices lie at lattice points. Since $DA = AR$ and $DT = TR$, the perimeter is $2(DA + DT)$. The distance from D (-10, -2) to A (-6, -4) is

$$\sqrt{(-10 + 6)^2 + (-2 + 4)^2} = \sqrt{20} = 2\sqrt{5},$$

and the distance from D to T (-6, -10) is

$$\sqrt{(-10 + 6)^2 + (-2 + 10)^2} = \sqrt{80} = 4\sqrt{5}.$$

The perimeter of the figure is $12\sqrt{5}$, or about 26.83.



8. 2.5. Both graphs have the same axis of symmetry: $x = -6$. Graph $f(x)$ and plot the vertex of $g(x)$ at (-6, -10). Let D and R be the unknown points of intersection of the two graphs, and let A and T be the vertices of the two graphs. Then $\overline{DR}$ and $\overline{AT}$ are the diagonals of the dart, and the area is $(1/2)DR \cdot AT = (1/2)DR \cdot 6 = 18 \rightarrow DR = 6$. The distance, then, from D or R to the line $x = -6$ must be 3, so the x-coordinate of R is -3. Then $f(-3) = |-3 + 6|/2 - 4 = -2.5$, the y-coordinate of R. The final step is to find the slope of $\overline{TR}$:

$$\frac{-10 + 2.5}{-6 + 3} = \frac{7.5}{3} = 2.5 = k$$

9. Since $\sqrt{3} < 2$, both expressions are positive real numbers. If their squares are equal, then the two numbers are equal. The square of the first expression is $2 - \sqrt{3}$. Squaring the expression $(\sqrt{6} - \sqrt{2})/2$ yields

$$\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)^2 = \frac{6 - 2\sqrt{12} + 2}{4}$$

$$= \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3}.$$

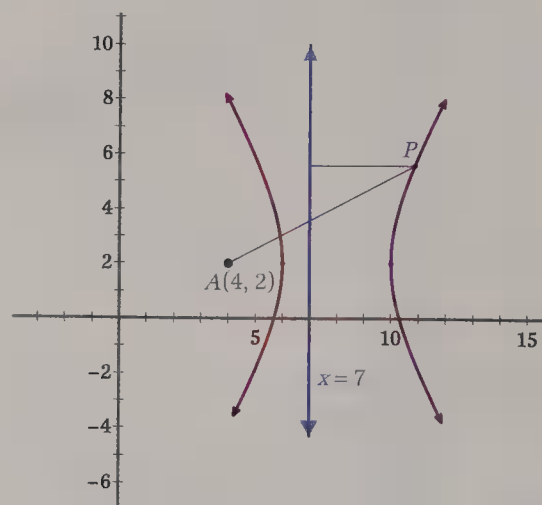
10. A hyperbola, centered at (8, 2) with vertices at (10, 2) and (6, 2). Write an expression for the distance from a point (x, y) to the point A (4, 2):

$$\sqrt{(x - 4)^2 + (y - 2)^2}$$

The distance from a point (x, y) to the line $x = 7$ is $|x - 7|$. Set up an equation, square both sides, and organize the resulting terms in a format that indicates the shape of the relation. We have the following:

$$\begin{aligned} \sqrt{(x - 4)^2 + (y - 2)^2} &= 2|x - 7| \\ x^2 - 8x + 16 + (y - 2)^2 &= 4(x - 7)^2 \\ -4(x^2 - 14x + 49) &= 0 \\ -3(x^2 - 16x + 60) + (y - 2)^2 &= 0 \\ -3(x - 8)^2 + (y - 2)^2 &= -12 \\ \frac{(x - 8)^2}{4} - \frac{(y - 2)^2}{12} &= 1 \end{aligned}$$

So the locus of points is a hyperbola centered at (8, 2) with vertices at (10, 2) and (6, 2).



11. $(\sqrt{6} - \sqrt{2})/2$. Use the 30-60-90° relationships to obtain $CB = 1/2$ and $AC = \sqrt{3}/2$. Since AD is 1, CD must be

$$1 - \frac{\sqrt{3}}{2} = \frac{2 - \sqrt{3}}{2}.$$

Apply the Pythagorean theorem to $\triangle BCD$:

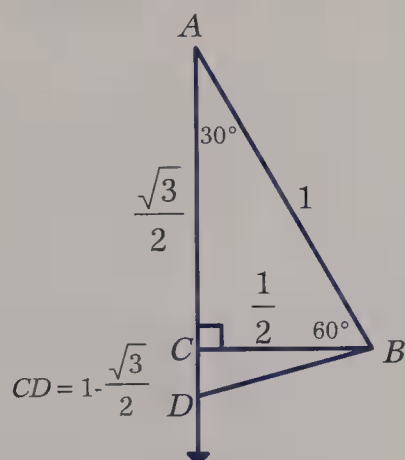
$$\left(\frac{2 - \sqrt{3}}{2}\right)^2 + (1/2)^2 = BD^2$$

$$\frac{4 - 4\sqrt{3} + 3}{4} + \frac{1}{4} = BD^2$$

$$2 - \sqrt{3} = BD^2$$

$$\sqrt{2 - \sqrt{3}} = BD$$

Use the result of the problem of March 9 to rewrite $\sqrt{2 - \sqrt{3}}$ as $(\sqrt{6} - \sqrt{2})/2$.



12. 66. We shall investigate $(a + b + c)^{10}$ as the expansion of the binomial $(a + b) + c$. We have $(a + b)^{10} + 10(a + b)^9c + 45(a + b)^8c^2 + 120(a + b)^7c^3 + \dots + c^{10}$. Neither the binomial coefficients nor the factor c^n will affect the number of terms. The changing powers of c give us one way to establish the fact that there will be no like terms to combine after the 10 terms containing $(a + b)^n$, $1 \leq n \leq 10$, have been expanded. The expansion of $(a + b)^{10}$ yields 11 terms, the expansion of $(a + b)^9$ yields 10 terms, and so on. So the total number of terms is $11 + 10 + 9 + \dots + 1 = 12 \cdot 11/2 = 66$ terms.

Alternate solutions: See solutions to the problems for March 19 and March 20.

13. 27. If we generalize the result of March 12, we find that $(a + b + c)^n$ will

have $(n + 2)(n + 1)/2$ terms. If $(n + 2) \cdot (n + 1)/2 = 406$, then the product of two consecutive integers is 812. Since $30^2 = 900$ and the units digit of $9 \cdot 8$ is 2, we guess $29 \cdot 28$ and obtain 812, so $n = 27$.

14. $4\pi/3$. Ignoring the thickness of materials used to construct these shapes, the cylinder inside the cube will have no effect on the size of the sphere. If the cube has surface area 8, then each face has area $4/3$, and an edge has length $2\sqrt{3}/3$. So the sphere will have radius $\sqrt{3}/3$, and surface area $4\pi(3/9) = 4\pi/3$.

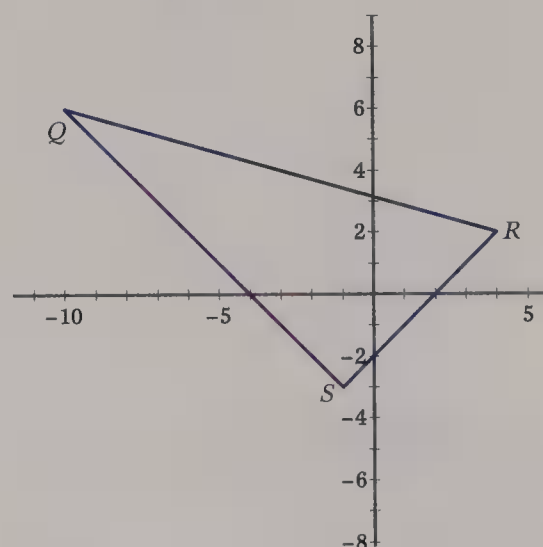
15. 180 units². When we connect the midpoints of the sides of any triangle, we create a triangle similar to the original triangle with $1/4$ the area of the original triangle. Plot the given points and label as shown. The plot shows us that $\triangle QRS$ is a right triangle with $QS \perp RS$. We find that

$$RS = \sqrt{(4 + 1)^2 + (2 + 3)^2} = 5\sqrt{2}$$

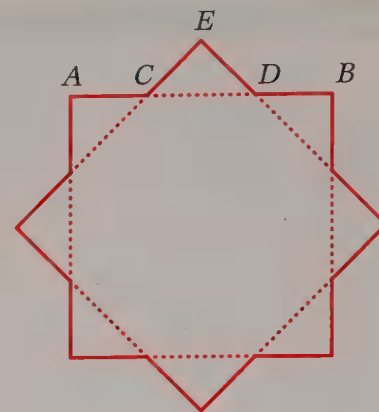
and that

$$QS = \sqrt{(-10 + 1)^2 + (6 + 3)^2} = 9\sqrt{2}.$$

The area of $\triangle QRS$ is $5\sqrt{2} \cdot 9\sqrt{2}/2 = 45$. So $\triangle ABC$ has area $4 \cdot 45 = 180$ units².



16. $128(2 - \sqrt{2}) \approx 75.0$ in. The perimeter of the 16-gon consists of the legs of eight congruent, isosceles right triangles. A side of the original square can be viewed as two legs and one hypotenuse: $AB = AC + CD + DB$. Let $AC = x$. Then $16 = 2x + x\sqrt{2} \rightarrow x = 8(2 - \sqrt{2})$. The perimeter of the 16-gon is $16x = 128(2 - \sqrt{2}) \approx 75.0$ in. (see the diagram at top of col. 3).



17. 272. Since $17a$ is a perfect square, let $a = 17b^2$. Now $17 + 17b^2 = 17(1 + b^2)$ has to be a perfect square, so $1 + b^2 = 17$ results in the smallest value to satisfy the problem. Thus, $b^2 = 16$. Finally, $a = 17b^2 = 17 \cdot 16 = 272$.

18. Problem of the Month. Readers are asked to encourage their students to submit solutions to this problem. Send solutions to Seán Madden (smadden1@greeleyschool.org) or Ricardo Diaz (Ricardo.diaz@unco.edu) for possible publication in a future issue of *MT*.

19. 35 ways. This question is equivalent to solving the equation $a + b + c + d = 4$, where a, b, c , and d are nonnegative integers less than or equal to 4. The ordered quadruplets (a, b, c, d) will represent the solutions. One approach is to make a list of all the possible quadruplets. Approaching the task systematically should prevent us from missing solutions. There are 4 distinguishable ways to arrange the integers 0, 0, 0, and 4, and there is only 1 distinguishable arrangement of the integers 1, 1, 1, and 1. To permute 0, 0, 1, and 3, because the 0s are indistinguishable, we have $(4!)/(2!) = 12$ distinguishable arrangements. There are likewise $(4!)/(2!) = 12$ distinguishable arrangements of 0, 1, 1, and 2. There are

$$\frac{4!}{2!2!} = 6$$

distinguishable arrangements of 2, 2, 0, and 0. We have a total of $4 + 1 + 12 + 12 + 6 = 35$ ways.

Alternate solution: This method is sometimes called the “stars and bars” method. Let the stars * * * * represent

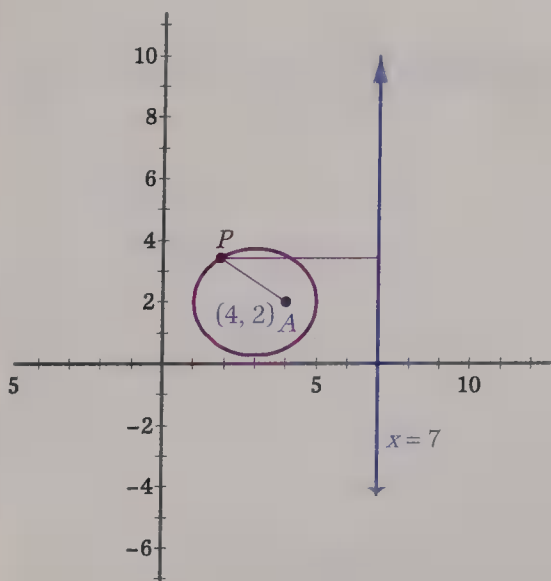
the 4 pieces of candy. We need 3 bars ||| to divide the stars into 4 groups. So one possible distribution of candy can be shown schematically as * * ||| * *, which corresponds to (2, 0, 0, 2). The schematic approach shows that we now have 7 symbols to arrange. Three of the symbols are bars, and 4 are stars. The number ${}_7C_3 = {}_7C_4 = 35$ gives us the answer we seek.

20. 35. Every term in the expansion will have the form $a^p b^q c^r d^s$, where p, q, r , and s are nonnegative integers less than or equal to 4 and $p + q + r + s = 4$. The solution to this problem is identical to the solution given for the problem of March 19. Note: The related problems 12, 13, 19, and 20 give specific examples of multinomial expansion. In the general case, where we wish to find the number of terms in the expansion of $(x_1 + x_2 + x_3 + \dots + x_m)^n$, we can use the formula

$$\frac{(n + (m - 1))!}{n!(m - 1)!}.$$

21. An ellipse centered at (3, 2). The reasoning used to solve the March 10 problem can be used again here. Note the position of the factor 2:

$$\begin{aligned} 2\sqrt{(x - 4)^2 + (y - 2)^2} &= |x - 7| \rightarrow \\ 4(x^2 - 8x + 16 + (y - 2)^2) & \\ - (x^2 - 14x + 49) &= 0 \rightarrow \\ 3x^2 - 18x + 15 + 4(y - 2)^2 &= 0 \rightarrow \\ 3(x - 3)^2 + 4(y - 2)^2 &= 12 \rightarrow \\ \frac{(x - 3)^2}{4} + \frac{(y - 2)^2}{3} &= 1 \end{aligned}$$



22. 98, 868. The smallest palindrome larger than 2014 is 2112. So 98 is the smallest integer that creates a palindrome when added to 2014. The smallest palindrome that creates a palindrome when added to 2014 must be greater than 98. The integer 99 does not work, so we look for a three-digit palindrome. The only way to have a 2 in the units place to match the 2 in 2014 is to place an 8 in the units place, implying that the leading digit will also be 8. So a possible solution has the form 8_8, and $2014 + 8_8$ gives us 28_2. The sum must be a palindrome, and 2882 works if the palindrome added to 2014 is 868.

23. No. The table shows all possible outcomes, or the sample space. The probability that the yellow die shows an even number is $P(A) = 1/2$. The probability that the sum of the faces is less than 6 is $P(B) = 10/36$. The probability that both events occur is $P(A \cap B) = 4/36$. But the product, $P(A) \cdot P(B) = (1/2)(5/18) = 5/36$. Since $P(A \cap B) \neq P(A) \cdot P(B)$, the events are not independent.

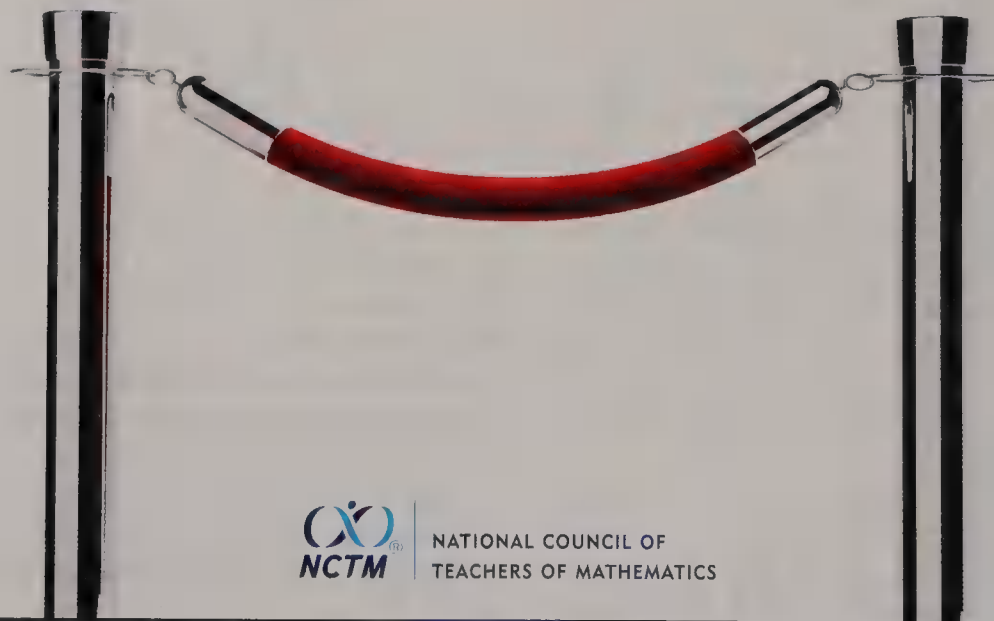
		yellow die					
		1	2	3	4	5	6
blue die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Alternate solution: If the two events are independent, then $P(A|B) = P(A)$. The expression $P(A|B)$ means “the probability of the yellow die showing an even number, given that the sum of the faces is less than 6.” So we look only at the reduced sample space inside the red lines on the table. In 4 of those 10 sample points, the yellow die shows even: $P(A|B) = 4/10$. Since $P(A|B) \neq P(A)$, the events are not independent.

24. $x^{-2}, 1/\sqrt{x}$. If $x = -5$, then $|x + 4| - |x| = 1 - 5 < 0$. If $x = -2$, then $|x + 2| - |x - 2| = 0 - 4 < 0$. The expression x^{-2} can be rewritten as $(1/x)^2$, and the square of any nonzero real number is

I ♥ hyperbolic cosines.

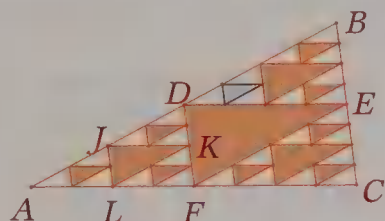
MATHEMATICS
IS ALL AROUND US.



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positive. For the final expression, note that the domain of x is real numbers greater than zero, so all defined values of the expression are positive.

- 25.** $37/64$. One small triangle has been highlighted in the figure. Triangle ABC can be subdivided into 64 congruent triangles of this size. Count 27 such triangles that are not shaded. Therefore, the requested probability is $37/64$.



- 26.** 4,989,600. There are 11 letters in the word MATHEMATICS and three pairs of letters. The 2 Ms, the 2 As, and the 2 Ts are indistinguishable. The number of distinguishable permutations is therefore

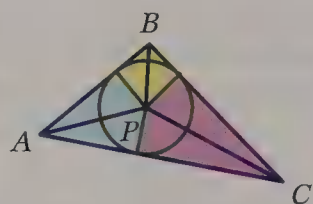
$$\frac{11!}{2!2!2!} = 4,989,600.$$

- 27.** 18 quarters. Let q = number of quarters and let n = number of nickels. Then $0.25q + 0.05n = 6.2$, and $0.05q + 0.25n = 9.4$. Multiply the first equation by 5 and subtract the second equation from it: $1.2q = 21.6 \rightarrow q = 18$.

- 28.** 496 and 8128. Solve the system: $16a + 28 = b - 164 \rightarrow b = 16a + 192$. Substitute into the second equation: $(16a + 192)/127 = (a - 112)/6 \rightarrow 16a + 192 = (127/6)(a - 112) \rightarrow 96a + 192 \cdot 6 = 127a - 127 \cdot 112 \rightarrow 31a = 15376 \rightarrow a = 496, b = 8128$. Since $496 = 2^4 \cdot 31$, the sum of the divisors of 496 is $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$. The factorization of 8128 is $2^6 \cdot 127$, so the sum of the divisors is $1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064 = 8128$.

- 29.** $Area = rs$, where r = radius of inscribed circle and s = half the perimeter of the original triangle. When the six small triangles are arranged to form a single rectangle, we note that the radius of the inscribed circle is one side of the

rectangle and that half the triangle's perimeter is the other dimension.



- 30.** $\frac{2 + \sqrt{3}}{1}$, or about 3.73:1. Find the

scale factor by comparing the length FH to $F'H'$. Let a side of the square $ABCD$ equal 1. Then the altitude of $\triangle FAB$ is $\sqrt{3}/2$, and $FH = 1 + \sqrt{3}$. Next, find the distance from H' to the midpoint of $\overline{AB}$:

$$1 - \frac{\sqrt{3}}{2} = \frac{2 - \sqrt{3}}{2}$$

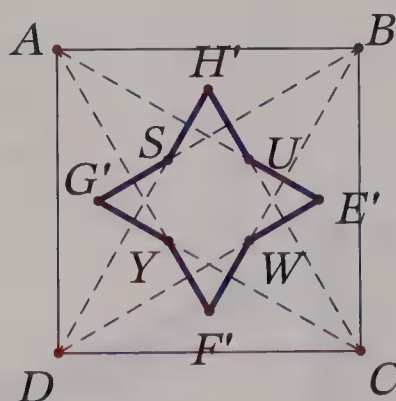
Therefore, $F'H'$ must be

$$1 - 2\left(\frac{2 - \sqrt{3}}{2}\right) = \sqrt{3} - 1.$$

Thus, the scale factor, larger to smaller, is

$$\frac{1 + \sqrt{3}}{\sqrt{3} - 1} = \frac{2 + \sqrt{3}}{1},$$

or about 3.73:1.



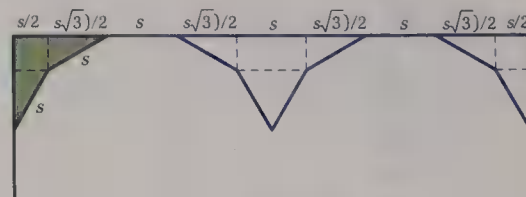
- 31.** Approximately 113 ft.². The apothem of each dodecagon is 6 ft. We can find the length of a side using trigonometry. If s = side length, then $0.5s = 6 \cdot \tan 15^\circ$. The perimeter of the dodecagon is $12s = 144 \cdot \tan 15^\circ$. Using the formula for the area of a regular polygon, $A = ap/2$ yields $A = (6 \cdot 144 \cdot \tan 15^\circ)/2 = 432 \cdot \tan 15^\circ$. There are 4 slabs of the same area: $1728 \cdot \tan 15^\circ \approx 463$ ft.². Subtract

from 24^2 to obtain the planted area: approximately 113 ft.².

Alternate solution: We can obtain an exact expression for the area without trigonometry. Doing so makes the demiregular tessellation—dodecagons, squares, and equilateral triangles—more obvious. The measure of each angle of a regular dodecagon is 150° . Therefore, the angle measures of the central nonconvex octagon (containing plants) are 210° and 60° . So this octagon can be partitioned as shown into four equilateral triangles surrounding a square. Each corner of the patio has a planted region that is $1/4$ of the central octagon, or two 30 - 60 - 90° right triangles and a square whose side is the short leg of the triangle. Let s be the side length of the dodecagon. The 30 - 60 - 90° triangle has hypotenuse s , short leg $s/2$, and long leg $s\sqrt{3}/2$. Its area is $s^2\sqrt{3}/8$. The square has area $s^2/4$. The area of the corner planting is

$$\frac{s^2(1 + \sqrt{3})}{4}.$$

Multiply by 16 to obtain the total area planted: $4s^2(1 + \sqrt{3})$. We observe that the 24 ft. length of the patio is $2s\sqrt{3} + 4s$. So $s = 24 - 12\sqrt{3}$, and $s^2 = 1008 - 576\sqrt{3}$. Substitute this value in the expression for the area and simplify to obtain an exact value: $1728\sqrt{3} - 2880$, or approximately 113 ft.², which is slightly less than 20% of the total patio area.





2015 Focus Issue

Creating Classroom Communities

NCTM guides mathematics teachers toward equitable teaching by emphasizing the importance of classroom communities. The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, and education researchers to share their experiences in building classroom communities. We encourage submissions that will help *MT* readers learn new ways to capitalize on the strengths that cultural, racial, and linguistic diversity bring to our classrooms and schools.

Classroom communities embrace individuals.

- How can we foster a sense of belonging in our classrooms?
- How can we learn about our students—their interests, issues that might be important to them, their languages, and their racial or ethnic communities?
- How can we incorporate this knowledge in our lessons and assessments? What are examples of effective tasks that highlight strengths of individual students? How can we balance individual, cooperative, and whole-class activities?

Classroom communities foster communication.

- How do we organize lessons so that students can share their mathematical ideas or solutions?
- What classroom norms are effective in facilitating communication?
- How can we encourage students to listen to, critique, and build on other students' mathematical thinking?
- How can we communicate high standards for students?
- How can technologies foster student communication about mathematics?
- What strategies are successful in removing barriers to student participation and engagement in mathematics?

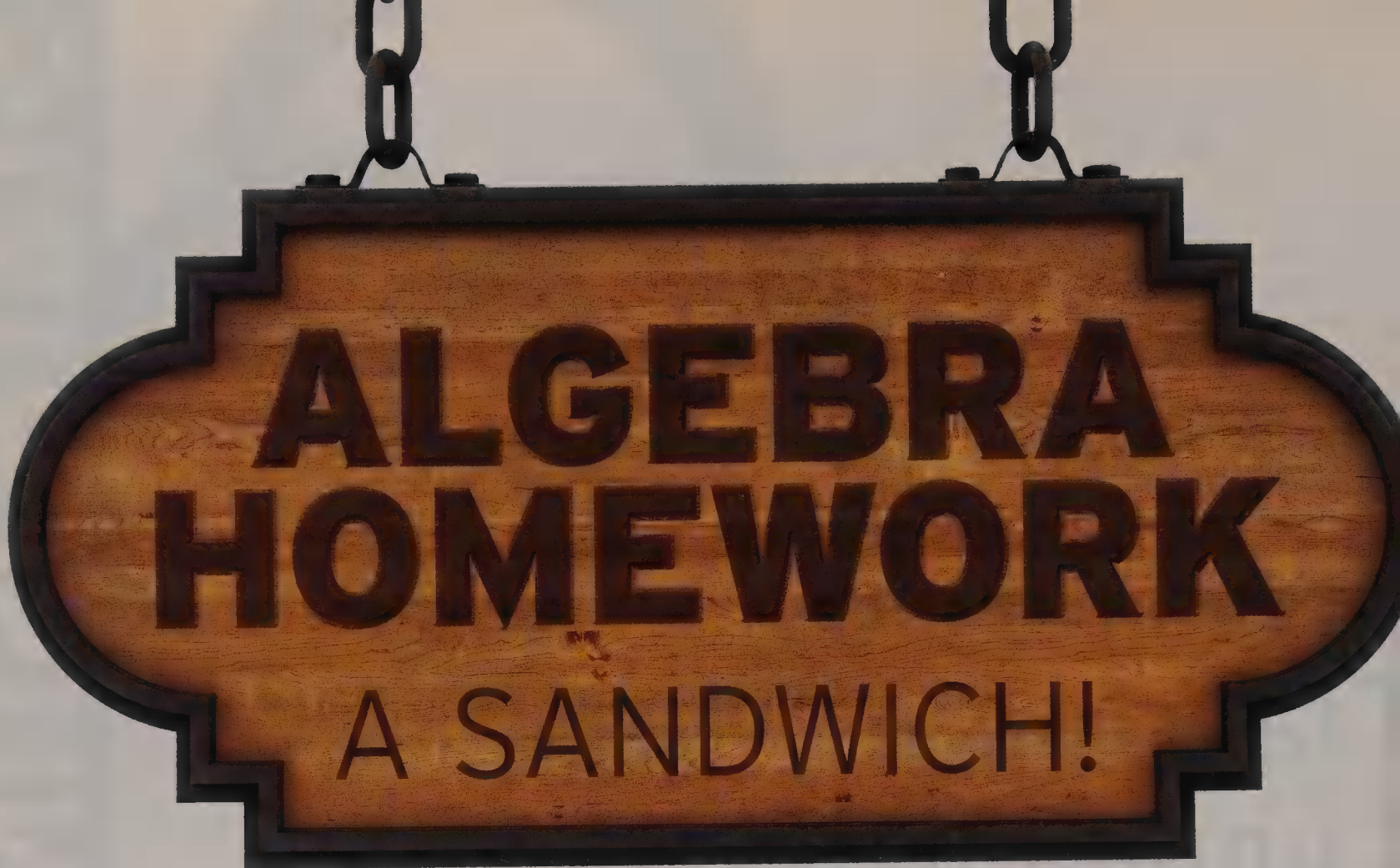
Please submit manuscripts at mt.msubmit.net by **May 1, 2014**. Be sure to enter the call's title (Creating Classroom Communities) in the Department/Calls field. No author identification should appear in the text of the manuscript. See www.nctm.org/publications/content.aspx?id=22602 for manuscript guidelines. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Tara Slesar, tslesar@nctm.org.



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CALL FOR MANUSCRIPTS



*Given two slices of bread—a problem
and the answer—students fill in the fixings:
their own mathematics reasoning.*

D. Bruce Jackson

Homework is a sandwich. I give you the problem and the answer, so there's no credit for these. What's inside the sandwich is your job—make it hefty and tasty!

— Marlo Warburton

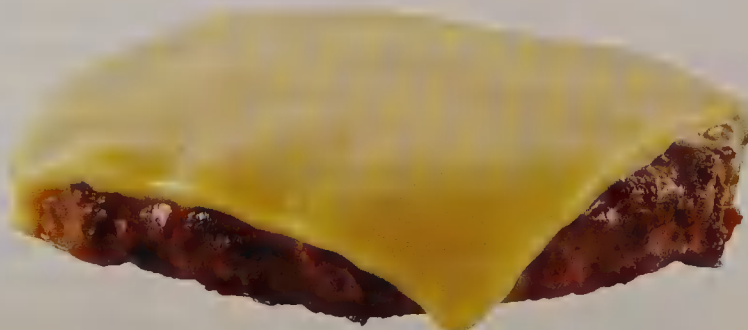
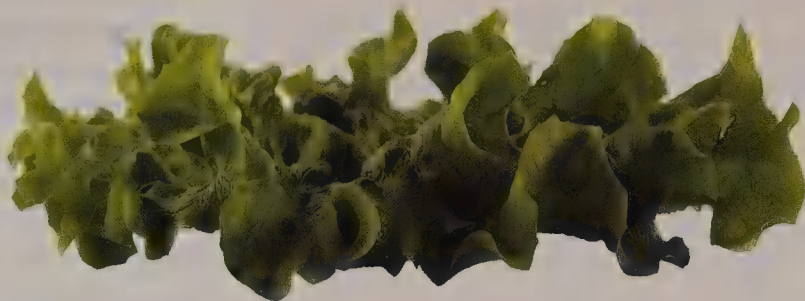
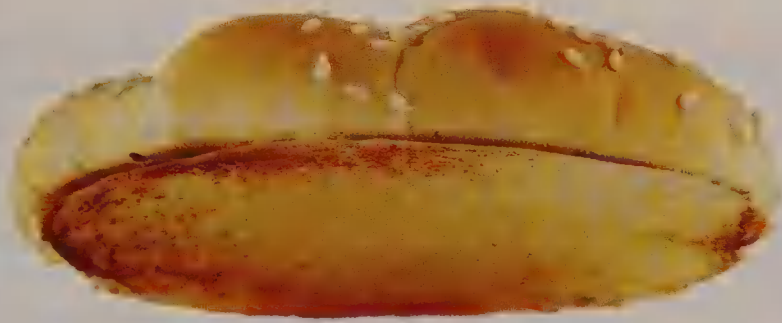
In Marlo Warburton's algebra classroom, homework is not about finding answers, which are all in the back of the book. It's about learning the mathematics to get those answers. It's also about recognizing when you really don't understand and need a little help.

At the start of the school year, students learn that what counts in algebra homework is quality of work, not quantity. They will face half as many problems as usual—ten instead of twenty to thirty. The problems will be a mix of skill, fluency, conceptual understanding, and applications problems; almost all will be odd-numbered—with answers in the back. But students will be expected to do every one of them correctly, checking their own work. If

by class time they still cannot figure some out, they will get a “partial credit” stamp on their homework calendar.

No stigma is attached to these partial credit stamps, however. “Score and fix” and “Learn from your mistakes”—these are the mantras for the first class sessions each fall as students review their papers in class. All must do their initial work in pencil and use red pens for corrections (see **fig. 1**). They practice locating answers, using sticky notes to bookmark the text's answer section.

Once students transition to full self-checking, they must have their papers out at the start of class, all problems completed and marked with red stars or Xs (**fig. 2** shows a sample of student-corrected work). As the teacher goes around the room recording homework completion on monthly homework calendars, students self-assess, with some saying that they still do not quite understand the problems and do not deserve full credit.



Don't Just Do Your Homework, LEARN it!

Step 1: Carefully read and do a homework problem in pencil.

Step 2: Check your answer in the back of the book. (If it's not there, mark "NIB".)

Step 3: Using red pen, score your work.

★★ GOT IT ON THE 1ST TRY

I got the problem right the first time I tried. I completely understand the problem.

★ GOT IT ON THE 2ND TRY

I got the problem wrong, did the problem over in red pen, and understand my mistake. I understand the problem now. You can see my original pencil work and red corrections.

✗ STILL DON'T GET IT

I tried my best, but the book says my answer is wrong. I can't find my mistake. I will get help on this problem from a teacher, tutor, friend, or family member.

Step 4: Repeat this process for each homework problem.

Step 5: At the top of your homework, write a fraction showing how many of the problems you understand (1st or 2nd try both count) out of how many problems were assigned.

Fig. 1 This handout is used to explain the homework system to students.

#1-13 ODD #41-51 ODD #71 Feb. 8, 2015

7-7

1. $7a^2 - 10a^2 + 9a$
 $-3a^2 + 9a$

2. $0.7r^4 + 32r^3 + .19r^4$
 $.26r^4 + .32r^3$

3. $5b^3c + bc^3 - 3b^3c$
 $3b^3c$

4. $(5n^3 + 3n^3 + 6) + (18n^3 + 9)$
 $8n^3 + 6 + 18n^3 + 9$
 $26n^3 + 15$

5. $(-3x + 12) + (9x^2 + 2x - 18)$
 $-3x + 12 + 9x^2 + 2x - 18$
 $-1x - 6 + 9x^2$

6. $(6c^4 + 8c + 6) - (2c^4)$
 $6c^4 + 8c + 6 - 2c^4$
 $4c^4 + 8c + 6$

7. $(2r + 5) - (5r - 6)$
 $2r + 5 - 5r - 6$
 $-3r - 1$

8. $(5u^2 + 3u + 7) - (u^3 + 2u^2 + 1)$
 $5u^2 + 3u + 7 - u^3 + 2u^2 + 1$
 $7u^2 + 3u + 8 - u^3$

9. $2(2x + 3) + 2(3x + 7) = 3c$

10. $3x + 6$
 $6x + 14$
 $2x^2 + 1x + 5$
 $(-5z^2)^2$
 $(-3z^{12})$

11. $9z^{12}$

12. $(5n^3 + 3n^3 + 6) + (18n^3 + 9)$
 $5n^3 + 3n^3 + 6 + 18n^3 + 9$
 $26n^3 + 15$

13. $(2r + 5) - (5r - 6)$
 $2r + 5 - 5r - 6$
 $-3r - 1$

14. $(5u^2 + 3u + 7) - (u^3 + 2u^2 + 1)$
 $5u^2 + 3u + 7 - u^3 + 2u^2 + 1$
 $7u^2 + 3u + 8 - u^3$

15. $2(2x + 3) + 2(3x + 7) = 3c$

Fig. 2 In this student-corrected homework paper, most problems were done correctly. For some, however, the student had forgotten to "distribute the negative" when subtracting one polynomial in parentheses from another.

If enough students seem to be having difficulties, Ms. Warburton will review strategies for solving the type of problem at issue, but students themselves remain responsible for their own solutions. They have two weeks or until the month's end to earn a homework upgrade to full credit, getting help with troublesome problems in the after-school tutoring sessions, which are held twice weekly and are well attended.

Periodically, students self-evaluate the homework process using a simple rubric (see fig. 3). Most say that they like the system and that it works well for them. Each fall a few ask to be given credit for "all that work" that they put into incorrect solutions and complain about the "extra work" of doing them over correctly. Ms. Warburton's response: "It's not 'practice makes perfect'—wrong practice is worse than none! *Perfect practice* is what really makes perfect."

In a midyear homework reflection, some 64 percent of students reported following the self-check procedures. More than 80 percent said that having answers helped them learn. As one student explained, "If you're doing it wrong and catch it, you don't have a whole section to do over." Another added, "If it's right, it encourages you to keep going." By the end of a year, complaints are rare, and the majority of students score 80 percent or higher on homework; in 2012 the median was 96 percent.

The Longfellow Middle School algebra program, codeveloped by Ms. Warburton and her colleague Juliana Jones, is rapidly becoming a legend in the district. Each year its students have outscored district and state averages by significant margins. U.S. Education Secretary Arne Duncan and Congresswoman Barbara Lee paid a visit in 2010 to honor the Longfellow program for outstanding achievement in mathematics and the impressive percentage of students of color performing at or above grade-level.

Since then, things have only improved. Last year, while other schools in the district moved their lowest-performing eighth graders into prealgebra, Longfellow continued to place the full spectrum of students into algebra. Even with these struggling students, Longfellow outscored the others by a substantial margin, with 75 percent achieving "proficient" or "advanced" ratings (see fig. 4). Longfellow eighth graders consistently list algebra as among their "best-ever" classes and openly strive for these ratings—this in a diverse Title I school with more than 50 percent of students receiving free or reduced-cost lunch.

Clearly, homework is part of a bigger picture involving excellent teaching, skillful classroom management, warm relationships with students,

and a carefully nurtured social environment that makes mathematics achievement a positive student norm. But the homework process is a critical piece of Ms. Warburton’s overall system, communicating a psychologically empowering level of trust and confidence in students’ ability to self-correct and continuously improve.

In my own role as a two-day-a-week volunteer tutor in the program, I see an impressive level of student-to-student help and support. Most remarkable is the quality of that interaction, which focuses on the process for arriving at answers. Students are not thinking about the Common Core Standards for Mathematical Practice, but they are certainly helping one another “make sense of problems and persevere in solving them” and learning to “understand the approaches of others in solving complex problems” (Standard 1, CCSSI 2010, p. 6). Similarly, by comparing their answers with those in the text, they are “learning to monitor and evaluate their progress and change course if necessary.”

Further enhancing the quality of the student exchanges is the teacher’s endorsement of multiple paths to a solution. There is only one right answer, but there are many right ways to attain it, so learning from a neighbor does not have to mean copying his or her exact procedure. In solving the simple equation $2(x + 2) = 16$, for example, it is equally acceptable to divide first or to distribute first. Equations, tables of values, and graphs are all acceptable routes to a solution in word problems involving initial fees (e.g., entry or membership) plus a per-unit rate (of time, rides, etc.). During daily mathematics warm-ups, typically involving types of problems currently being studied, students first work alone, then compare approaches in five minutes of collaborative time, and finally take part in a whole-class discussion of the alternatives.

“Hooray for mistakes!” Mistakes are considered a normal and valuable part of a learning process that has its inevitable bumps and setbacks. What is important is how students respond to frustrations and what they learn as they correct errors.

This message is reinforced through the integral connection between homework and testing procedures. Quizzes in alternate weeks are composed entirely of problems from recent homework. Any major discrepancy between quiz scores and homework scores is an occasion for student self-reflection: “Am I learning the mathematics behind the homework or just getting it done?”

Following every test is a class assignment to correct (in red pen) any solutions that received less than full credit. Students can raise a poor test grade up to a 70 (a C– but still proficient) by coming in after school and working with tutors or other students to correct the missed problems. Once students

Reflecting on last night's algebra homework . . .

- Did I check my answers in the back of the book?
- Did I check each answer immediately after each problem?
- Did I catch any mistakes?
- Was I able to figure out what I did wrong?
- Did I fix my work (not just the answer) in red pen?
- Are there any problems I still don't understand?
- What is my plan for learning those problems?
(Tutoring, Family/Friend, No Plan, Learned Already)
- Did I write my final score as a fraction at the top of the page?
- Did last night's homework answers help me learn?
- In general, does having homework answers help me learn?
Comments?

Fig. 3 This rubric can serve as a template for students' daily homework reflection.

Longfellow Eighth Graders Scoring "Proficient" or "Advanced"	
Year	Percentage
2006	10
2007	22
2008	44
2009	40
2010	64
2011	67
2012	75

Fig. 4 This chart shows the approximate percentage of Longfellow eighth graders scoring “proficient” or “advanced” on the state algebra test over a seven-year period.

have shown understanding of the issues involved, they take a new version of the test to confirm their understanding for the record. Students know that one bad performance will not derail them for the whole term.

By accepting mistakes and catching them early, students know that they are not practicing something wrong that they will have to unlearn later. They are also absorbing larger lessons about perseverance and are building a positive mindset linking achievement to personal effort.

WHY IT MATTERS

Why do all these details matter? First, most teacher credential programs offer little or no guidance to prospective mathematics teachers about how to deal with such seemingly nuts-and-bolts matters as homework. Anyone Googling “algebra I homework policy” will read plaintive pleas from novices overwhelmed with evaluating and providing feedback on homework, “advice from experience” about not accepting late homework, or warnings against

assigning odd-numbered problems (“It just invites cheating” or “Students should do those on their own if they need help in figuring something out”). Actual policies, now often posted online, range across a wide spectrum from permissive to punitive. Textbook publishers offer little homework guidance, and some mathematics texts provide answers in a separate teachers-only answer booklet. Mathematics teachers, meanwhile, often see no viable alternative to spending precious class time going over the previous night’s homework, even when those students who most need the homework practice are not completing enough of it to benefit from the review.

Algebra occupies a unique border territory within mathematics’ subcultures and hierarchies, alternately claimed or shunned by both middle school and high school mathematics departments. It is a region where student-centered and subject-centered teaching approaches frequently confront each other under conditions of unequal status, divergent beliefs, and limited communication.

High schools are often home to a toughness subculture that sees rigidity as rigor and equates strict grading policies with high standards. Many high school teachers whom I have met are convinced that middle school teachers coddle their students and that algebra in eighth grade must be a watered-down version. “This is the real world,” they insist, “No late homework is accepted; no retests are allowed.” (Never mind the real-world reality that lateness on taxes or bills carries only modest penalties or that most licensing tests can be taken repeatedly.)

Reinforcing the toughness ideology is a residual belief that mathematics ability is innate and in short supply: “You either have it, or you don’t.” Algebra teachers have often seen themselves as gatekeepers, making sure that those who “don’t have it” are not passed along to certain failure in advanced mathematics classes. Assigning homework with answers known in advance clashes head-on with this belief system.

The consequences are substantial. Since the earliest days of Bob Moses’s Algebra Project (see Moses and Cobb 2001), algebra has been recognized as a key gateway to higher education and economic success, a crucial focus of efforts to narrow the racial and socioeconomic achievement gap. Algebra homework has the potential to widen that gap sharply when more advantaged students successfully complete it, whereas the less advantaged enter a downward spiral of unfinished work, failed tests, negative attitudes, and shrinking effort—what Goldberg (2007) calls the “homework trap.” But unless in-school mathematics time is greatly expanded at the expense of other subjects, the extra practice and self-assessing of homework remain critical parts of mastering algebra, particularly for students with relatively weak mathematics backgrounds.

So it is essential that algebra teachers design homework systems that encourage the kinds of behaviors and attitudes that lead to long-run success in mathematics. If they fail, the students for whom homework is most essential will be those least likely to complete it.

CHANGING THE HOMEWORK PARADIGM

At a time when the debate continues over whether homework is overused, optional, or essential or favors well-off students over those with little home support, we must understand ways in which effective homework strategies can help narrow the achievement gap. Vatterott (2009, p. 94) argues convincingly that the “old paradigm” of difficult, grade-affecting homework “creates practical and motivational obstacles that converge to form the perfect storm for student failure” for those who do not fit the traditional mold of well-prepared student and supportive family. To make a homework system productive for all students, it must encourage mathematical thinking and skill development; provide useful, nonjudgmental feedback; enhance student responsibility for learning; help students self-assess without advertising weakness; and promote student self-confidence and motivation for learning.

The homework-as-sandwich system does all these things. In much of what educators must currently do in this standardized test-dominated era, answers have greater weight than the



INDIGOLOOTOS/THINKSTOCK

thinking behind them. Machine-scored multiple-choice tests weigh a lucky guess the same as a laboriously worked-out answer. This is not the case in Ms. Warburton's classes. With both problem and answer given, completing homework is a bit like solving a maze. Students know where to start and where to end, so they try promising mathematics paths until one works. No one snickers when a student takes a wrong turn or retraces steps. Feedback is prompt and private—gone is the fear that tomorrow's homework check will reveal an embarrassing misstep in a student's thinking. Not being able to figure something out is an expected part of learning (see **fig. 1**); instead of hoping that no one will notice, students can take the initiative and ask for help. Once students learn each successful problem-solving approach, they earn full credit for that learning. There are no "gotchas" lurking in algebra class—it's a friendly place.

CONCERNS, QUIBBLES, AND CAVEATS

What about all those concerns about rigor and high standards or possible cheating? Let's start with cheating. Cheating is possible in any system in which students complete work beyond the direct supervision of a teacher, but it is most likely to occur when the stakes are high, workload is heavy, and students doubt their own ability to do the work correctly. In the homework-as-sandwich system, stakes are low because work can be redone for full credit and homework performance is reported in a work habits grade, separate from the algebra proficiency grade. The workload is reasonable, and students get prompt and frequent affirmation of their ability to solve problems independently.

Standards in the homework-as-sandwich system algebra classes are amazingly high. Most students achieve real proficiency, and even those who do not still make remarkable progress. They are never let off the hook for learning, as is common in "rigorous" classes where failing a test or falling behind can lead students to withdraw effort completely. I have heard high school students say, "What's the use? I'll just take the class over again." One of my former students, taking algebra for the fourth time, had so internalized his sense of impending failure that he could not believe that he might pass without cheating.

Students in Ms. Warburton's classroom have not experienced as many failures, and she is determined to ensure that they will not. The incentives for attending tutoring sessions range from the warm herbal teas and positive environment to exhortations in the classroom and at parent conferences. Even students who know in late May that they will be retaking algebra in ninth grade show up for tutoring, work hard, and complete homework.

Is this a perfect system? Not quite, and in any case it is still evolving—this year there is more emphasis on mindset coaching plus consideration of increased weight for cumulative tests that show mastery of material missed on earlier tests. But in its present form, the homework-as-sandwich system surpasses the many alternatives I have seen in a long educational career.

Last spring, a newly minted mathematics PhD from nearby University of California at Berkeley was visiting Ms. Warburton's after-school "tea and tutoring" session, and I asked about her experience in college mathematics courses. "The classes where I learned the most were ones where I had answers to the problem sets," was her response. After all, at the upper reaches of mathematical thinking, brilliant insights often come where the answers are known, but the steps for attaining them must be logically demonstrated. Why can't this process begin more often in algebra class?

ACKNOWLEDGMENTS

I wish to thank Marlo Warburton and her Longfellow Middle School students, who model the vision of effective algebra homework presented in this article. Ms. Warburton, marlowarburton@berkeley.net, is a Math for America Master Teacher at the University of California at Berkeley.

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LEARNING ALGEBRA FROM WORKED EXAMPLES

Presenting examples of both correctly and incorrectly worked solutions is a practical classroom strategy that helps students counter misconceptions about algebra.

Karin E. Lange, Julie L. Booth, and Kristie J. Newton

“Make sense of . . .”

“Reason quantitatively . . .”

“Critique the reasoning of others . . .”

These phrases will sound familiar to anyone who has internalized the next stage of standards-based mathematics education—the Common

Core State Standards, currently being implemented in more than forty states.

With its focus on both content and process standards, the Common Core State Standards Initiative (CCSSI 2010) seeks to deepen students’ understanding in mathematics. Although the Common Core standards were written to address all grades, the goal of increasing comprehension is especially critical in first-year algebra, which functions as a gate-keeper course to higher mathematics and education (Star and Rittle-Johnson 2009).

Solving equations is a specifically pivotal unit in beginning algebra, one that is often confusing and challenging for many students. Many misconceptions that prohibit students from developing a deeper understanding of algebra have occurred and

persisted since their early learning experiences. For example, misconceptions surrounding the generalizability of addition or the meaning of the equals sign that may begin in elementary school negatively affect students’ understanding of variables and solving equations in beginning algebra classes (Samo 2009; Warren 2003). As a result of an increased focus on reasoning and sense making and less emphasis on procedural repetition (NCTM 2000) as well as the adoption of the Common Core State Standards (CCSSI 2010), algebra teachers may need additional tools to correct commonly held misconceptions. Most important, traditional instruction, even when designed to address misconceptions, often does not provide for sufficient conceptual change in student understanding (Vlassis 2004).

For students to be successful in algebra, they must have a truly conceptual understanding of key algebraic features as well as the procedural skills to complete a problem. One strategy to correct students’ misconceptions combines the use of worked example problems in the classroom with student self-explanation. *Self-explanation* is the “activity of generating explanations to oneself” (Chi 2000, p. 164), especially “in attempt to make sense of new information” (p. 163) as one reads or studies.

A *worked example problem*, to be differentiated from *working an example problem*, shows students an already completed problem and directs their attention to certain steps of the task as the focus of questioning. Self-explanation, then, specifically encourages students to identify the reasoning behind the steps that they see carried out and to explain why these steps were completed. This strategy of providing worked example problems coupled with prompts for self-explanation has recently been shown to influence students' learning positively in both traditional (Booth, Koedinger, and Paré-Blagoev 2011) and computer-based classrooms (Booth et al. 2013). In addition, the Institute for Education Sciences promotes worked examples as a worthwhile instructional practice (Pashler et al. 2007), and several studies have shown that the technique has promise in classroom settings (Ward and Sweller 1990; Schwonke et al. 2009).

A unique and powerful aspect of using worked examples in the classroom occurs with the inclusion of examples of both correct and incorrect solutions (subsequently referred to as correct and incorrect examples). Some educators may fear that exposing students to incorrect examples may increase misconceptions, but research has shown otherwise (Booth et al. 2013). Using incorrect examples forces students to think about the steps that have been carried out and the reasons why these actions are wrong and then to confront their own possible underlying misconceptions. The desired result is a deeper understanding of mathematics for all students, regardless of prior skill level.

LINKING TO THE COMMON CORE

Solving equations is a key unit in beginning algebra, and the Common Core State Standards (2010) address this content area under the following two standards:

A-REI.1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

A-REI.2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. (p. 65)

With this phrasing, the Common Core Standards suggest that solving equations alone is not sufficient; students must demonstrate a deeper understanding of the concepts underlying the process of equation solving. When using the strategies suggested here, students must do more than simply

view worked examples; rather, they should engage in reasoning and discussions about the examples. Self-explanation becomes an additional tool for teachers to encourage students to “make sense of problems,” “reason abstractly and quantitatively,” “construct viable arguments,” and “critique the reasoning of others” (CCSSI 2010, pp. 6–7).

By using probing questions that require students to explain a previously worked example, teachers can ensure that students are making sense of what solving equations really entails. Students also engage in reasoning while constructing explanations and strengthen critical thinking skills while critiquing the correct or incorrect solutions. These tasks, carried out in conjunction with the Common Core State Standards, serve to promote deeper understanding of solving equations, which will help students of all ability levels prepare for higher-level mathematics.

USING EXAMPLES IN A COMPUTER-BASED ACTIVITY

Correct and incorrect examples have been used successfully in a classroom setting through a computer program, Cognitive Tutor, that responds to students' ability by giving more or less practice according to their patterns of answering questions. Students receive immediate feedback about their choice of explanations. The answer turns green if the student responds correctly. If the student answers incorrectly, the answer turns red, and a pop-up box appears with a suggestion to review certain aspects of the problem. This feedback has been thought to increase the effectiveness of this type of exercise. In this computer-based classroom setting, worked example problems were incorporated by replacing a procedural problem in Cognitive Tutor.

One exercise using a correctly worked example to address a misconception began like this:

Jose was asked to solve the following equation for x , and he made a *good* first step to solve the problem. Look at his first step:

$$\begin{aligned} 4/x &= -7 \\ 4 &= -7x \end{aligned}$$

Isolating this step, students might consider the significance of variables in the denominator and performing the same operation on both sides of the equation. Students were then prompted to answer two questions: “What did Jose do?” and “Why is that a *good* step for Jose to take?” The latter requires a two-part answer explaining both why the step was valid and why it was helpful. Students answered each of the three questions on the computer, using multiple drop-down boxes to construct complete answers. Given multiple options to choose

Table 1 Sample Constructed Answers about Jose's Worked Problem		
What did Jose do?	Why is that a <i>good</i> step for Jose to take?	
	<i>It is valid because . . .</i>	<i>It is helpful because . . .</i>
He multiplied both sides by x .*	It combines like terms that are constant.	It put all the constant terms together.
He added x to both sides.	It combines like terms that are variable terms.	It isolated the variable.
He divided both sides by x .	It does the same operation to both sides.*	It moves the variables out of the denominator.*

*Correct answers

Table 2 Sample Constructed Answers about Laura's Worked Problem		
What did Laura do?	Why is that a <i>wrong</i> step for Laura to take?	
	<i>It is invalid because . . .</i>	<i>It is valid but not helpful because . . .</i>
She added 3 to the right side.	It combines terms that are not like terms.*	It didn't reduce the number of terms.
She added -3 to $5x$.*	It performs the operation to only one side of the equation.	It made the numbers more complicated.
She subtracted 3 from $5x$.*	It performs different operations to both sides of the equation.	

*Correct answers

from in each drop-down box, students had available a total of 180 complete sentences for the first question alone. Sample answers that students could construct are provided in **table 1**.

Students were explicitly told when a worked example was incorrect, as in this instance:

Laura was asked to solve the following equation for x , and she made a *wrong* first step to solve the problem. Look at her first step:

$$2 = 5x - 3$$

$$2 = 2x$$

This type of example helps students address misconceptions about combining like terms. Again, two questions followed: "What did Laura do?" and "Why is that a *wrong* step for Laura to take?" In this case, the second question can be approached in one of two ways: "It is invalid because . . ." or "It is valid but not helpful because . . ." Hundreds of sentence combinations could be created; a few sample answers for this question appear in **table 2**.

One benefit of using Cognitive Tutor is that it allows for a controlled research environment in an actual classroom. Results from a study of these types of example problems (Booth et al. 2013) show that students perform better overall on an equation-solving posttest after receiving both correctly worked and incorrectly worked examples (rather

than receiving only correctly worked examples). The posttest included both procedural and conceptual measures of growth, and the study also found that students improved their conceptual knowledge the most after they had seen both the correct and incorrect examples. Contrary to the position that students with little prior knowledge should be shielded from incorrect examples, results showed that students who performed poorly on the pretest and then analyzed only the incorrect examples showed the most gain in conceptual understanding.

USING EXAMPLES IN A TRADITIONAL CLASSROOM

Another way to incorporate this approach in a traditional classroom is through student homework. Even without the immediate feedback of a computer-based system, students displayed better conceptual understanding after completing assignments that included analyzing correctly worked and incorrectly worked examples. In one ongoing project, alternative versions of homework worksheets were created in which half the practice problems were replaced with worked examples for students to explain, and each example was followed by a corresponding problem that students would solve independently. A mix of clearly marked correct and incorrect solutions were provided and accompanied by prompts asking the students to explain something about the example. These prompts were

Example	Matched Problem
$6 - k = -3$ <u>Correct Example</u>  Eliza solved this problem correctly. Here are the steps she used to solve the problem: $\begin{array}{r} 6 - k = -3 \\ -6 \quad -6 \\ \hline -k = -9 \\ -1 \quad -1 \\ \hline k = 9 \end{array}$ a) Why did Eliza subtract 6 from both sides? b) Why did Eliza divide both sides by -1?	$-6 - k = 3$

Fig. 1 Questions prompt students to analyze a correctly worked example.

Example	Matched Problem
$3 + 6x = 4 - 5x$ <u>Incorrect Example</u>  Jackson tried to solve this problem, but he didn't do it correctly. Here is his first step to solve the problem: $\begin{array}{r} 3 + 6x = 4 - 5x \\ 9x = 4 - 5x \end{array}$ a) Which terms did Jackson combine to incorrectly get $9x$? b) Give an example of two terms that would correctly sum to $9x$.	$-6x + 3 = 4 - 5x$

Fig. 2 Questions prompt students to analyze an incorrectly worked example.

designed to target specific common misconceptions and call students' attention to key conceptual features of the problem.

Consider the example in **figure 1**. Students are asked to look at Eliza's solution and prompted to think about why Eliza's steps were appropriate. Then, at the right, students face a matched problem to solve on their own. In **figure 2**, students are presented with Jackson's flawed attempt to solve the problem and then asked questions about what was done incorrectly. Finally, students face a similar problem to solve independently. These problems continue to target common misconceptions by focusing student attention on the negative signs and combining like terms.

Student work is shown in **figure 3**.

This approach was not limited to use in solving equations. Worked examples and justification were also used to address misconceptions regarding word problems, problems with graphs, and number lines (see **fig. 4**).

THE STUDENT EXPERIENCE

Teachers' initial hesitation to using both correct and incorrect examples may stem from concerns

$$3 + 6x = 4 - 5x$$

Incorrect Example

Jackson tried to solve this problem, but he didn't do it correctly. Here is his first step to solve the problem:

$$3 + 6x = 4 - 5x$$

$$9x = 4 - 5x$$

Which terms did Jackson combine to incorrectly get $9x$?

3 and $6x$

Give an example of two terms that would correctly sum to $9x$.

$$3x + 6x$$

(a)

$$6k = 3$$

Correct Example

Hannah solved this problem correctly. Here are the steps she used to solve the problem:

$$\begin{array}{r} 6k = 3 \\ \hline k = \frac{1}{2} \end{array}$$

Why couldn't Hannah just subtract 6 to get the k by itself?

Because in the problem she is dividing k and so she has to do the inverse operation of division which is multiplication.

(b)

Fig. 3 Students analyze mathematical reasoning.

that students will have extra work or be confused by the incorrect examples. When students were first shown the example exercises, they often expressed confusion, saying things like "What is this . . . ?" or "I don't know what to do." However, as students moved on with additional worked examples, their comments changed. One student reflected, "I'm not really sure what she did . . ." before going on to answer the problem correctly. Another student exclaimed, "Oh, it makes me think!" During the exercises, students were actively critiquing the reasoning of others. For instance, in response to Laura's worked example, one student explained, "She subtracted, like, $5x$ minus $3x$, but she shouldn't have."

When introducing a new strategy, teachers may face resistance from students. Such was the case with students who had completed units incorporating the worked example problems shown here. In the classrooms participating in the study detailed by Booth et al. (2013), students were first surprised by the new approach but then acknowledged that they were actually learning more. Sixty-two percent of students reported liking the module, although one student noted, "It was hard to do the 'find what's wrong' [problems]." However, 82 percent of students reported believing that the module was helping them learn. Sample responses from the student survey included these: "It made you do math"; "It taught me new ways to look at things"; "It helped me to learn things a little better"; and "Well, it helped when I got back to taking the posttest because I think I got more answers correct." In brief, although students found the process somewhat challenging, they reported that it helped them learn.

TARGETING STUDENT MISCONCEPTIONS

Traditional mathematics instruction in solving equations often involves the teacher demonstrating the necessary steps to solve for the variable. Even when probing questions are asked or exploratory activities are completed, students may not be prompted to challenge their misconceptions explicitly. Incorporating worked example problems into the classroom offers a way for students to confront their misconceptions and to replace them with a deeper conceptual understanding. The most growth in student learning was found in a classroom environment in which students consider both correct and incorrect examples (Booth et al. 2013). In light of the Common Core push for reasoning skills, it is imperative that educators have access to strategies to promote conceptual understanding.

Results from the two research studies—Booth, Koedinger, and Paré-Blagoev (2011) and Booth et al. (2013)—suggest that for students to correct misconceptions, they need to see more worked examples, engage in discussion or explanation of these examples, and critique the reasoning behind incorrect mathematical statements. Worked examples here have been introduced through computer-based and homework-based methods. Teachers may be able to replicate these results by adapting the key components of this strategy—focused questions and student discussion on already completed examples—in a variety of ways.

Coupling both correctly and incorrectly worked examples with self-explanation allows students to confront their own misconceptions. Unlike other methods for remediating student difficulties, this type of intervention is beneficial for students at all

Stephan works as a waiter and made \$40 in tips each night he worked this week. He gave \$25 of his tips to the serving assistant and had \$215 left for himself. How many days did he work this week?

Correct Example



Melinda set up this problem correctly. Here is what she wrote:

$$40x - 25 = 215$$

- What does x stand for in Melinda's equation?
- How did she know to subtract 25 rather than add it?

(a)

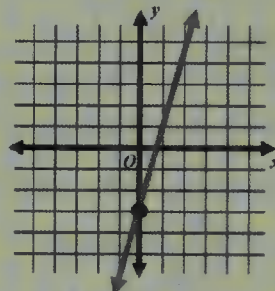
The line contains the point (2,5) and has a slope of 4.



Correct Example

Andrew solved this problem correctly. Here is how he solved the problem:

$$\begin{aligned} y &= mx + b \\ 5 &= 4(2) + b \\ 5 &= 8 + b \\ 5 - 8 &= 8 + b - 8 \\ -3 &= b \\ y &= 4x - 3 \end{aligned}$$



- How did Andrew know that (0, -3) was the y-intercept, not the x-intercept?
- How could Andrew have checked whether he graphed the line correctly?

(b)

$$-4 \geq x$$



Incorrect Example

Tenyim tried to graph the inequality, but he didn't do it correctly. Above is his graph.

- What did Tenyim do wrong when graphing the inequality?
- What solution is not represented in this graph?
- How could Tenyim represent this solution?

(c)

Fig. 4 Understanding word problems and graphs can also be improved through analyzing worked examples.

ability levels, a key factor when working toward student achievement in a beginning algebra course. This approach affords all students the opportunity to explore mathematical reasoning and build a sound conceptual understanding of algebra.

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MATHEMATICS
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Working the Sy

Two teachers discuss the planning and observed results of an introductory problem to help students nail a conceptual approach to solving systems of equations.

Darla R. Berks and Amber N. Vlasnik

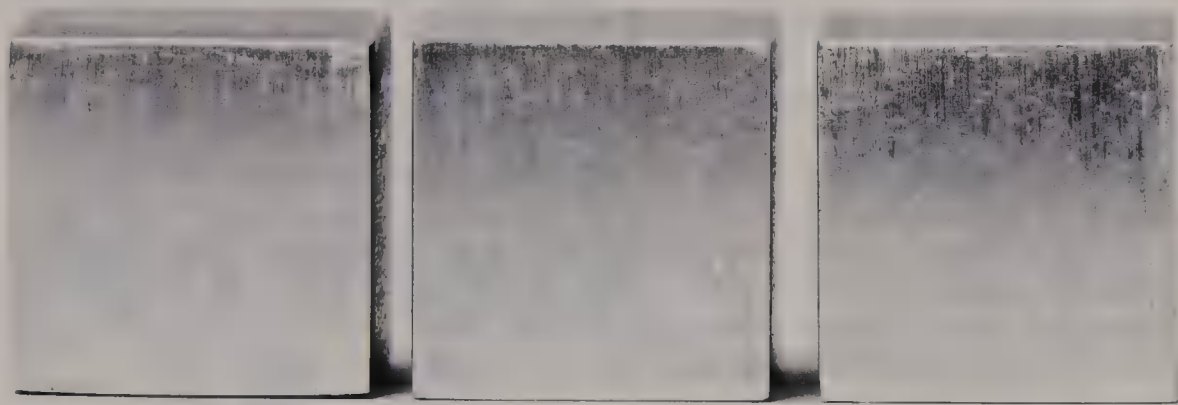
“**C**lass, today we will learn method 1 for solving systems of linear equations—graphing. Tomorrow we will learn method 2—substitution. And then we will learn the third and final method—elimination.”

Does this introduction to a unit on systems of equations sound familiar?

Unfortunately, many of us learned about the concept of systems of linear equations in this procedural way. The lessons were taught as three discrete methods. Connections between the methods, in many cases, were not made. As a result, the students' overall understanding of the concept was very limited. By the time we reached the end of the unit, students were so overwhelmed by these different methods that they struggled even to get started on the problems. They just looked for the procedure and often asked, “What method do I use on this system?”

We found ourselves asking why. Is there a better way to teach this unit so that students can start to make those connections themselves? Out of these frustrations came the Farmer Fred prob-





stem

lem and an entire lesson focused on his balancing dilemma. We will describe the reasoning and thinking that took place in this planning, explain this introductory problem, and discuss the benefits that we observed with our students.

PLANNING THE LESSON

While planning a unit for systems of linear equations, we wanted to accomplish several goals. Our overarching goal in all algebra lesson planning is to help students become better problem solvers and thinkers. We hope to increase students' ability to persevere when presented with a challenging mathematical task. We also reflected on the Common Core Mathematics Standards, recognizing that the standards are a "balanced combination of procedure and understanding" (CCSSI 2010, p. 8), and wanted our unit to reflect that balance. We also know that effective mathematics lessons consist of tasks that are "cognitively demanding as well as pedagogical practices that are suitable to support collaboration and discourse among students, and thoughtful engagement with mathematical reasoning, problem solving, and explanation" (Silver 2010, p. 1).

Taking into account all these considerations, we decided to begin with a scenario that looked different from a typical mathematics problem. We did not want operation symbols or numbers to be a part of the initial situation presented to students. Our hope was that the absence of symbols and numbers would emphasize the thinking involved in the process and aid student understanding before adding in the computation and the procedures. Along with this thinking process, we knew that we wanted students to begin with a task in which they would be engaged, working with their peers and communicating their



reasoning. We wanted to be intentional about allowing students to verbalize and explain their thoughts as well as evaluate and critique the thinking and reasoning of others.

THE FARMER FRED PROBLEM

We imagined that this nonnumber introduction would gradually change throughout the course of the lesson to include numbers, variables, and, finally, two complete equations. We knew that we wanted to make frequent connections throughout the unit between the different aspects of learning systems of equations—not only connections among the three methods of solving a system of equations but also connections with the variety of ways (tables, graphs, and equations) to represent the system and interpret the solution.

During our brainstorming, it quickly became apparent that the algebraic manipulation of equations is a major piece of what students have traditionally experienced while learning these concepts. As Proulx and colleagues noted in their work on teaching systems of equations, “Our research suggests that . . . the algebraic presentation of systems of equations heavily predominates” (Proulx et al. 2009, p. 529). Looking at our textbook and observing classroom presentation, we too found this observation to be true. Rather than simply tell students that they were learning substitution, we decided to create a nonnumber introduction with a situation in which students think algebraically without using numbers or variables. The concept of substitution would then evolve much more intuitively.

Our initial problem presented to students used the following information:

Farmer Fred was using a balance and noticed that these combinations of boxes, washers, and nails were equal. How many nails would balance one washer? How many nails would balance one box? (See **fig. 1**.)

Students were given time to think about this problem and write down their ideas and conclusions. At this point, students were asked to explain their thinking and the process they used to answer the questions, first in writing and then verbally with their peers. We then refined the Farmer Fred problem by introducing numbers (see **fig. 2**).

Now, rather than balancing weight, students were asked to assign values to the items. By this point, students were naturally beginning to do so on their own. Students were asked: “What is the value of one box? What is the value of one washer?” To help students continue to move from pictures to equations, we asked them to rewrite these equations using numbers and variables.

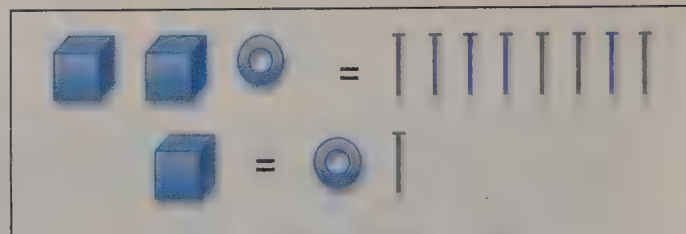


Fig. 1 Boxes, washers, and nails are balanced.

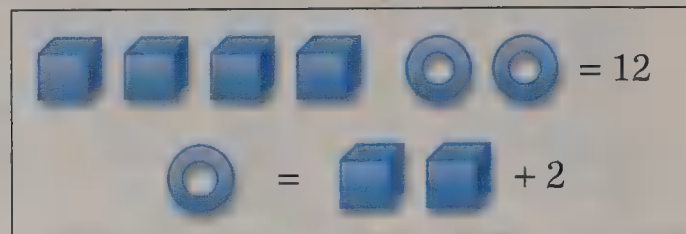


Fig. 2 Students naturally assign values when numbers and operation symbols are introduced.

Students would write $4x + 2y = 12$ and $y = 2x + 2$. Because they had already been engaging in mathematical discussions about Farmer Fred and his dilemma, this abstraction was not too much of a stretch for them. They found that the variables, rather than being distractions, led to a rich discussion in which many connections were made.

THE MATHEMATICAL DISCOURSE AND CONNECTIONS

Discussions that focus on cognitively challenging mathematical tasks—namely, those that promote thinking, reasoning, and problem solving—are a primary mechanism for promoting conceptual understanding of mathematics (Smith et al. 2009, p. 548). We believed that the student discussions were helpful for understanding the underlying algebraic process. Even within this introductory lesson, students could already see the ideas of substitution and combining like terms. Our students drew on their experience solving equations with Algebra Tiles™ to see that inverse operations were used to isolate a variable. They then realized that once the value for one variable was determined, they could find the value for the other variable as well. We would later build on this idea of needing to know one variable and using its value to solve for others as the method of elimination was introduced.

Another encouraging aspect of this discussion arose when the two equations were used to create tables of values (see **fig. 3**) and used to graph the lines (see **fig. 4**). Some students were not sure how to complete the table with the first equation $4x + 2y = 12$ in standard form; some changed the equation into slope-intercept form before completing the table. Although the process of creating the tables was difficult for some students, it became a powerful tool to confirm the validity of values that we claim to be solutions to a given system.

$4x + 2y = 12$		$y = 2x + 2$	
x	y	x	y
0	6	0	2
1	4	1	4
2	2	2	6
3	0	3	8
		4	10

Fig. 3 Tables show a solution common to both equations.

The step of creating the tables and graph was necessary and beneficial. It allowed students to notice in both of these representations that the solution was an ordered pair—an actual point common to the solution sets of both equations. Students coming into the class with a variety of educational backgrounds were able to use whichever representation made the most sense to them.

When students start to make connections among graphical, tabular, symbolic, and descriptive contexts for functional relationships—all representations of the same relationship—they may feel inspired and engaged by seeing structure and beauty among these representations (Middleton and Jansen 2011).

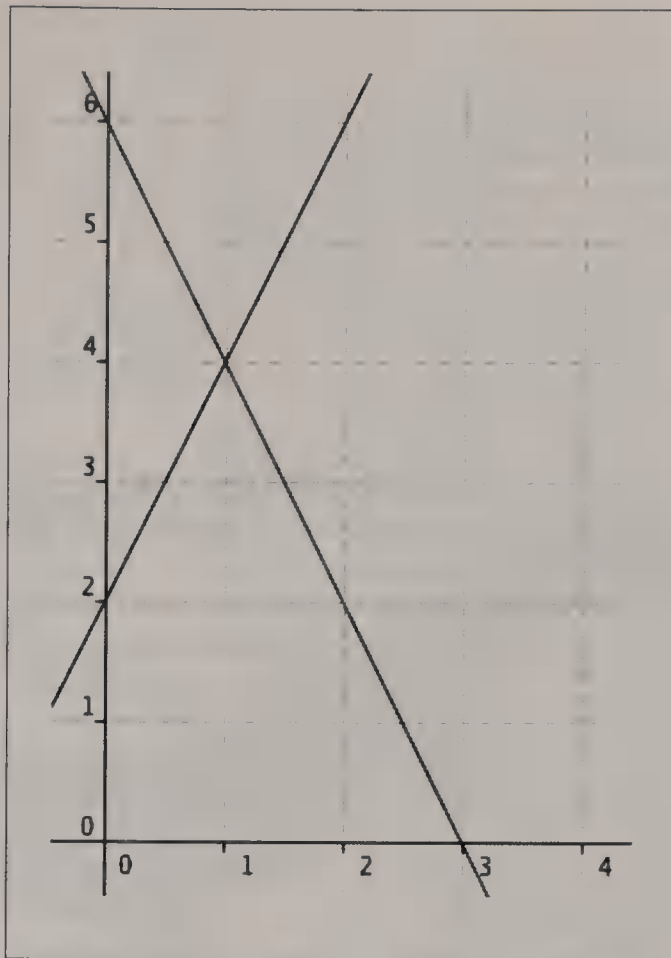


Fig. 4 The graphical intersection shows a solution common to both equations.

Technology Tips

Which technologies enrich your teaching? Which motivate your students?

- // Do you have a great way to use technology in your classroom? *Tell us how!*
- // Perhaps you have some reflections on technology that you would like to share. *What should we be thinking about?*
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OUTCOMES OF THE NEW APPROACH

Throughout this lesson, we observed many important outcomes for our students. First, we were reminded again of the extreme importance of conceptual engagement and productive struggle. Although getting students accustomed to this type of task takes time and intentionality, teaching conceptually is worth the work. In previous years, some students struggled to choose which variable to substitute. After this approach, our students had a good understanding of why substitution worked.

We also noticed that the progression of adding numbers and variables to the problem was beneficial. It gave the students a chance to think initially about what was happening and why. The numbers, symbols, and variables—which could have become stumbling blocks—instead served as useful steps to move students toward a correct solution. Although the final step of solving a system of algebraic equations with substitution was still challenging to some students, we could see the benefits of our approach. When students were a bit unsure about how to solve the system, we referred back to the Farmer Fred problem and followed the same process and thinking involved with that example. This transition was a key part in students' understanding.

Another benefit that we noticed was the mathematical discourse. We were actively listening to the students as they were explaining their thinking to their partners. We probed and prompted as needed, taking notes on what we saw and heard that could be used for whole-class discussion. Having listed the key ideas that we wanted to be a part of our whole-class discussion, we made quick notes of the names of students who had good or unique responses or had made drawings that could be used to explain those key ideas.

For example, in one class, we noted that Sarah had a great drawing showing that the two boxes each contained a washer and a nail. We used that drawing as a part of our whole-class discussion. We also noted that Nang had a great verbal explanation describing three washers and two nails on one side of the balance. The whole-class discussion continued in this way; it was primarily led by students, with our facilitation. The reasoning was part of the mathematical concept of the lesson. The fact that the explanations were coming from peers rather than solely from the teacher seemed to give them more merit in the eyes of the students.

Finally, we were reminded of the importance of multiple representations. Some students understood the mathematical concept with just the algebra involved. Others had an “aha” moment when they saw the graph of the intersecting lines, whereas still others truly grasped the concept when they saw the table of values. Having all these

representations seemed to give students a much more complete idea of what it means to solve a system of equations.

Watching our students grow and gain understanding throughout this unit of study was exciting. One seemingly simple problem about a farmer and his balance proved to be useful and versatile. The connections that students made among the methods used to solve systems of linear equations enabled them to make their own decisions when starting a problem they had never seen before. They were able to become problem solvers, in part because the abstraction within the problems was no longer intimidating—it had a purpose!

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Sarah I. Duncan, Suzanne Lenhart, and Kelly K. Sturner

Measuring Biodiversity with Probability



JAMES DEBOER/THINKSTOCK

More and more, teachers are asked to make connections between the STEM disciplines—science, technology, engineering, and mathematics—while also addressing state standards. Mathematics is an underappreciated but important tool for the life sciences, from mathematically modeling biological processes to making sense of biological data. The activity presented here was designed for a Girls in Science camp, held at Tremont, Tennessee, in the Great Smoky Mountains National Park. The camp is designed to give local girls entering eighth grade a chance to become familiar with the natural world by doing hands-on research in the park. This particular exercise was designed to show the value of mathematics for quantifying and interpreting biodiversity data that the girls had collected.

Biodiversity is important for ecosystem health and productivity. Greater biodiversity provides greater opportunities to find new organisms that can be important food sources or new medicines for people around the world. Also, high biodiversity makes ecosystems more resilient; with increasing threats to species from global climate change or natural catastrophes, more species can mean

a greater potential that some organisms will have the necessary traits to survive.

But what exactly is biodiversity? How do we go about measuring whether biodiversity in one place is greater than biodiversity in another? Scientists and ecologists do not simply count up all the different organisms and compare the totals. Measuring biodiversity involves applying probability to obtain a more meaningful index for comparison.

These two activities allow students to apply tools from mathematics to life science, raising awareness of connections between STEM disciplines. The activities were developed for grades 9–12 and will take approximately one hour each, depending on the students' level of experience with probability. Mathematics teachers using these activities are encouraged to communicate or work with biology teachers to strengthen the activity. If time allows, students could collect data and talk about biodiversity as part of their biology class exercise and complete the activity in a mathematics unit on counting or probability. Group work is also appropriate for this exercise to allow students to discuss difficult terminology and reduce the amount of class time the activity will take.

Activities for Students appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more information on the department and guidelines for submitting a manuscript, please visit <http://www.nctm.org/publications/content.aspx?id=10440#activities>.

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Activity 1 serves as an introduction to quantifying biodiversity and can be used alone for students in beginning high school mathematics courses. This activity will be most successful in a classroom in which students have a basic understanding of counting and probability. It also provides a real-world application for these skills. Students are introduced to Simpson's Diversity Index, which uses principles of probability to determine objectively which area has the greater biodiversity. After being led through a derivation of Simpson's Diversity Index, students then apply the equation to a data set on their own. At the end, students may discuss whether their results support their original hypotheses. Activity 1 provides an opportunity to learn and use knowledge connected with the following Common Core State Standards: seeing structure in expressions (algebra), building functions (functions, modeling), and conditional probability and rules of probability (statistics and probability, modeling) (CCSSI 2010).

For advanced classes (usually precalculus or higher), the module may be continued with activity 2, which requires an understanding of how to calculate limits. Students use mathematical practice skills from the Common Core State Standards—reasoning abstractly and quantitatively, constructing viable arguments, and using appropriate tools strategically (CCSSI 2010)—to discover how species richness and species evenness affect the index. Considering extreme cases of richness and evenness that may not occur in a biological scenario can aid students in understanding the constraints of this model.

This module may serve as an introduction to measuring biodiversity and then may be applied to a real-data set gathered by students. Students could, for example, count insects from two different areas on the school grounds or learn to inventory tree species found in their yards at home or in a local park.

First, students will learn key vocabulary and concepts. Teachers might begin by asking students for their definition of biodiversity. Already students may implicitly understand that what they are describing is a measurement and involves quantification. The sample data set found in **table 1**, tree species that were counted

Table 1 Number of Trees		
Species	Yard A	Yard B
Eastern redbud	3	5
Black oak	4	5
Post oak	5	5
White pine	3	5
Honey locust	1	5

in two yards, may be used to introduce and illustrate the key concepts of species richness and species evenness.

- *Biodiversity* is a measure of the different kinds of organisms (species) in a region or other defined area. Usually when quantifying biodiversity, we look at a group of similar organisms (such as all insects), rather than all the organisms in the ecosystem. This term takes into account both species richness and species evenness.
- *Species richness* is the number of unique species in a region or specified area. The data set shows five different species of trees found in the backyard (see **table 1**). Five is a measurement of the richness value.
- *Species evenness* is the degree of equitability in the distribution of individuals among species. Greater evenness signifies less variation in the numbers of individuals of each species. Maximum evenness occurs when the number of individuals among all species is the same. In **table 1**, we see that yard B has exactly the same number of each kind of tree and, thus, has greater species evenness than yard A.

Although richness makes intuitive sense for measuring biodiversity, students may wonder why species evenness would be important. Consider the example of a forest where introduction of a new species gives rise to a beetle infestation that causes population decline in other insect species. One additional species, adding to the richness value of insects but reducing the evenness, would thus have a negative impact on the biodiversity.

A useful next step might be to assess students' understanding of the concepts informally. Teachers can do so by presenting another table of data and having

students discuss species richness and evenness.

After students are comfortable with these concepts, ask them to imagine this scenario: An ecologist collects information from two separate experimental plots of the same size but with one big difference: Plot 1 is in the woods, and plot 2 is in a nearby field. The ecologist is interested in the types of insects that are found in the plots. The first data table in activity 1 is an example of the data that might be collected. Before being shown the table, students should think for a minute about what they expect to find and formulate a hypothesis.

A UNIVERSAL WAY TO DETERMINE BIODIVERSITY

Students might wonder which would be considered more biologically diverse—a plot with more richness or a plot with more evenness. What happens if one plot has many more species—would it be as easy to determine which plots had more evenness?

To answer such questions, in 1949 the British statistician Edward H. Simpson devised an objective method to measure biodiversity that is still used today. Simpson suggested that ecologists calculate overall diversity in terms of probability.

Simpson's Diversity Index can be explained as follows: If we randomly select an individual from a sample without replacement and then randomly select another individual from the same sample, what is the probability that the two organisms will be of different species? The higher the probability of different species, the higher the diversity. Simpson's Diversity Index, D , can be represented as

$$D = 1 - \left(\frac{n_1(n_1 - 1) + n_2(n_2 - 1) + \dots + n_s(n_s - 1)}{N(N - 1)} \right)$$

where S represents the number of species, n_i represents the number of individuals in the i th species, and N represents the total number of individuals. [The quantity $1 - D$ is also sometimes referred to as Simpson's Index, so careful reference to other resources is warranted. D as given here is also known as the Gini-Simpson Index.—Ed.]

MORE TO EXPLORE

Biodiversity can be quantified in more than one way. Consider having students do research on Shannon's Index (Shannon 1948), an alternative method of calculating a type of biodiversity index, and compare it with Simpson's Diversity Index. Also, Simpson's Diversity Index is not as useful when representing biodiversity across trophic levels (such as predators and prey). It does not make ecological sense to expect the same population sizes among predators and prey, if one predator needs to consume multiple prey species to survive. Ask students to think about ways to quantify total ecosystem biodiversity. There is no single right answer, but perhaps one student will publish a possible solution in a scientific journal someday.

For a small real-data set to explore, consider **table 2**, showing some of the species of salamanders found in the Great Smoky Mountains National Park. Thirty-one species can be found within the park's boundaries (National Parks Conservation Association, n.d.); thus, the Smokies are often called the salamander capital of the world.

MATHEMATICS AND BIOLOGY

Integration of mathematics with biology can illustrate that the concepts learned in mathematics class are not isolated exercises but are used in real-world applications that contribute to scientific knowledge. Students may realize that quantifying biodiversity could affect conservation efforts and ecosystem management. Biology-based exercises can provide an exciting and relevant way to reinforce mathematical concepts and explore the natural connections between mathematics and science.

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Table 2 Number of Salamanders at Selected Locations in the Great Smoky Mountains National Park

Species	Lower Dorsey Stream	Pig Pen Stream
Spotted dusky salamander	7	18
Imitator salamander	6	3
Seal salamander	5	15
Black-bellied salamander	7	11
<i>Desmognathus spp.</i> salamander	4	17
Blue Ridge two-lined salamander	1	31
Spring salamander	2	1
Northern slimy salamander	0	1
Black-chinned red salamander	0	0
Santeetlah salamander	1	0
Southern red-backed salamander	2	0

Note: These salamanders were identified in submerged bags filled with leaf litter from two different streams (Lower Dorsey and Pig Pen) in 2008. Data were collected as part of a citizen science program at the Great Smoky Mountains Institute at Tremont. After your students explore these data, have them look for more data on the Web or ask them to collect their own.



These activity sheets are available to teachers as Word documents that may be copied and edited for classroom use; go to www.nctm.org/mt. For solutions to the activities, download one of the free apps for your smartphone and then scan this tag to access www.nctm.org/mt052.



ACTIVITY 1: Behind Simpson's Diversity Index

Look at the data below and make a hypothesis. Do you think that biodiversity will be higher in the woods or in the field? Why? Can you suggest a way to decide by using mathematics?

Species	Plot 1: Woods	Plot 2: Field
Pillbug	50	10
Monarch butterfly	36	50
Seven-spotted lady beetle	35	0
Western honeybee	55	39

Simpson's Diversity Index was invented by a British statistician in 1949 and is based on probability. To see how it works, you can derive the equation for the index. We will start with a very simple case:

Imagine that you went into your back and front yards and counted the insects in each area. You found only two species of insects—the Western honeybee and the seven-spotted lady beetle (see chart below).

Insects	Back Yard	Front Yard	Totals
Western honeybee	10	35	
Seven-spotted lady beetle	50	40	
Totals			

You have totals of 60 individuals in the back yard and 75 individuals in the front yard.

- Using the back yard data:* Imagine that you put all those insects from the back yard into a bag. Suppose that you reach into the bag and randomly remove an individual and then randomly remove a second individual without replacement.
 - What is the probability that the two selected individuals are both Western honeybees?
 - What is the probability that the two selected individuals are both seven-spotted lady beetles?
 - What is the probability that the two individuals are from different species? Hint: This event is the complement of the union of the two events in questions 1(a) and 1(b).
- Using the front yard data:* Imagine that you put all those insects from the front yard into a bag. Suppose that you randomly select an individual and then randomly select another individual without replacement.
 - What is the probability that the two selected individuals are both Western honeybees?
 - What is the probability that the two selected individuals are both seven-spotted lady beetles?
 - What is the probability that the two selected individuals are from different species?

ACTIVITY 1: Behind Simpson's Diversity Index (continued)

3. Suppose that we now have n_1 of one species and n_2 of a second species in our area. The total number of individuals is now $N = n_1 + n_2$. See the chart below:

Species Type	Number of Individuals
Species 1	n_1
Species 2	n_2
Total Individuals	N

Suppose that you randomly select an individual and then randomly select another without replacement.

- (a) What is the probability that the two selected individuals are both species 1?
 - (b) What is the probability that the two selected individuals are both species 2?
 - (c) What is the probability that the two selected individuals are from different species?
4. Suppose that you have a very large data set, with many more than just 2 species. You might also have monarch butterflies, craneflies, praying mantises, and other insects, for a total of S species (see the chart below).

Species Type	Number of Individuals
Species 1	n_1
Species 2	n_2
Species 3	n_3
$\vdots$	$\vdots$
Species S	n_S
Total Individuals	N

Show that the probability that the two selected individuals are different species would be

$$D = 1 - \left(\frac{n_1(n_1 - 1) + n_2(n_2 - 1) + \cdots + n_S(n_S - 1)}{N(N - 1)} \right).$$

This is Simpson's Diversity Index, D . Hint: Use your reasoning from question 3(c).

5. If D is close to 1, is the area's biodiversity high or low? Justify your answer.
6. Calculate D for the plot 1 (woods) and plot 2 (field) data. Which plot has the higher D value? Does this calculation support your original hypothesis?
7. Could biodiversity be quantified in other ways? Justify your answer. Make suggestions about other ways to quantify biodiversity. Check the Web to find other indices that give alternative methods of calculating a biodiversity index.

ACTIVITY 2: Exploring the Effects of Species Richness on Simpson's Diversity Index

$$D = 1 - \left(\frac{n_1(n_1 - 1) + n_2(n_2 - 1) + \cdots + n_s(n_s - 1)}{N(N - 1)} \right)$$

$$= 1 - \frac{\sum_{i=1}^S n_i(n_i - 1)}{N(N - 1)}$$

Simpson's Diversity Index (D) takes into account both species evenness and species richness. *Species richness* measures the number of unique species in a region or specified area. *Species evenness* is the degree of equitability in the distribution of individuals among species. But just how important is each factor in determining D ? Can you get a very high index value with just one of these factors? In this activity, we will explore these questions to better understand how this mathematical model of biodiversity works. Note that the cases below with the same number of individuals in each species would not occur in nature. We are considering only the case of complete evenness.

1. Suppose a plot has four species with 100 individuals of each species (see the chart at right). Calculate D for this example.

Species Type	Number of Individuals
Species 1	100
Species 2	100
Species 3	100
Species 4	100
Total Individuals	400

2. Repeat exercise 1 to calculate D with 200 individuals in each species. Then calculate D with 500 individuals in each species. How do these answers compare with each other and with the answer from exercise 1? For $S = 4$, how does D change as we increase the total number of individuals but keep the proportions the same?
3. Suppose that a plot has four species and n individuals in each species. Calculate D for this example. Show that as we increase n but keep the population proportions the same, the values of D will decrease.
4. Suppose that a plot has S species with n individuals in each species. In this case, find D as a function of S and n and simplify as much as possible.
5. Using your result from exercise 4, show how D changes as S gets large and n remains fixed. Varying S allows us to see the effect of increasing species richness.

Find $\lim_{S \rightarrow \infty} (D(S, n))$.

6. Using your result from exercise 4, show how D changes as n gets large but as S remains fixed. Varying n as we have done previously allows us to see the effect of increasing the numbers of individuals when the population proportions stay constant.

Find $\lim_{n \rightarrow \infty} (D(S, n))$.

From the March 2014 issue of

MATHEMATICS
teacher

Grass for Goats (The Silo Version)

The Grazing Goat problem, familiar to many teachers and students, has several variations. The version presented here provides a rich opportunity for engaging students in a project spanning several weeks. Three solutions are discussed: one suitable for a calculus class, one suitable for a geometry class, and one suitable for a precalculus class. Although we start with a calculus approach, most of the article uses only algebra and geometry concepts. Also discussed are the didactics of using projects to open ever-larger fields of mathematics to students.

The problem may be stated as follows:

A retired mathematics professor has decided to raise a goat. He owns a silo and a barn. The barn's front wall is tangent to the silo at the corner (see **fig. 1**). The silo has a circular base with a radius of 10 feet. The professor has decided to tether the goat to a chain that is anchored at the corner of the barn, the point of tangency. He has also cut the chain so that it is long enough to wrap around the silo exactly once—that is, the length of the chain equals the circumference of the silo. Help the professor calculate the grazing area of the goat. Present a solution, complete with diagrams and calculations, explaining your answer. You are encouraged to use calculators and computers.

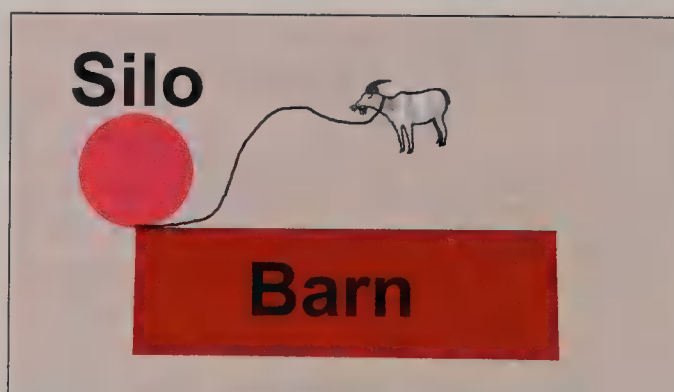


Fig. 1 At the barn's corner, the front wall is tangent to the silo.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more background information on the department and guidelines for submitting a manuscript, visit <http://www.nctm.org/publications/content.aspx?id=10440#delving>.

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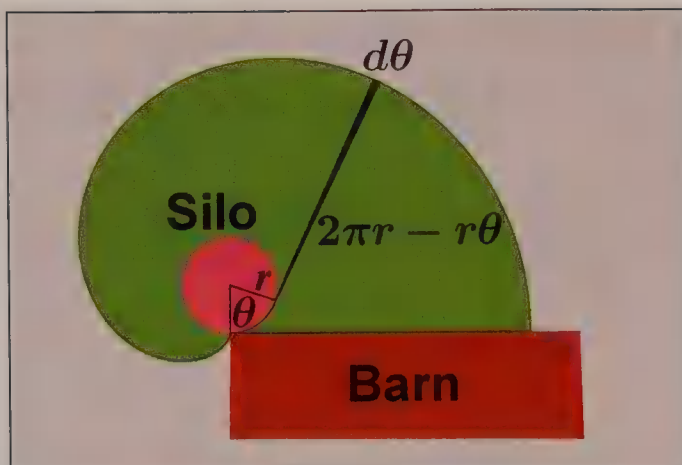


Fig. 2 The grazing area is limited by the length of the chain.

A CALCULUS APPROACH

To represent the region of grass, which I call the “grazing field,” we need to find the points of farthest reach of the chain. Each point on the curved boundary of the green field in **figure 2** is obtained by stretching the chain along a segment tangent to the silo. Along the perimeter of the silo, starting from the anchor point to the point of tangency, the chain subtends an arc of measure θ , so the arc has length $r\theta$. Since the chain has total length $2\pi r$, the tangent segment has length $2\pi r - r\theta$.

Hence, for each differential change in θ , the area swept out in the field is characterized by a differential circular sector of area $\pi(2\pi r - r\theta)^2 \cdot d\theta / (2\pi)$. Summing these differential sector areas for θ from 0 to 2π gives the area of the grazing field:

$$\begin{aligned} \int_0^{2\pi} \frac{1}{2} (2\pi r - r\theta)^2 d\theta &= \frac{1}{2} r^2 \int_0^{2\pi} (2\pi - \theta)^2 d\theta \\ &= \frac{1}{2} r^2 \left[4\pi^2 \theta - 2\pi \theta^2 + \frac{1}{3} \theta^3 \right]_0^{2\pi} \\ &= \frac{4}{3} r^2 \pi^3 \end{aligned}$$

Using calculus we can beautifully and efficiently solve this problem, but this approach is not within reach of most high school students. What would be a reasonable problem exercise within a calculus course can become an interesting project in a geometry, algebra, or precalculus course.

For projects, I like rich problems that have many solution strategies, leading students to make choices along the way. When given to geometry, algebra, or precalculus students, the Grazing Goat problem immediately presents choices to be made in the very representation of the problem. The two strategies I present for solving without calculus both approximate the area of the field by representing the silo as a regular polygon. We will start with a triangle, increase the number of sides, and look for a pattern.

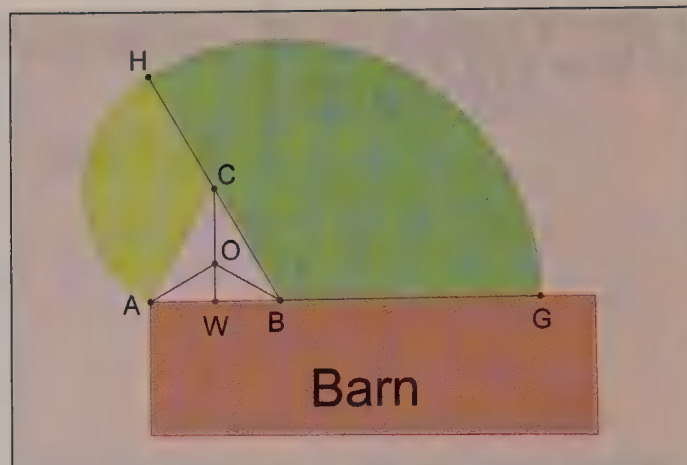


Fig. 3 A triangular silo with its near edge adjacent to the barn wall leads to a grazing field with two circular sectors.

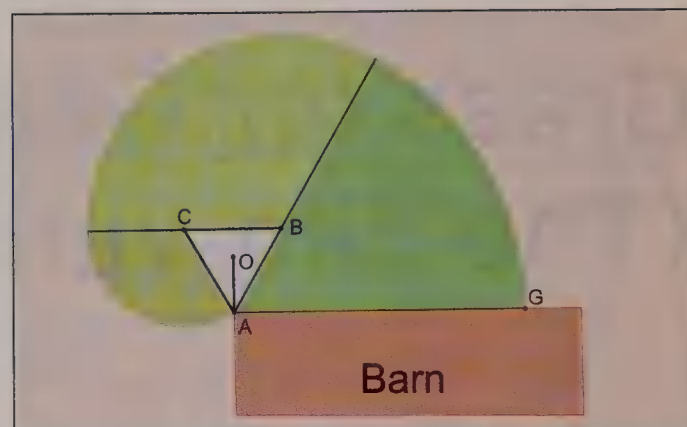


Fig. 4 A triangular silo with its far edge parallel to the barn yields a grazing field with three circular sectors.

CHOICES THAT STUDENTS MUST MAKE

One of the first choices that students must make to approximate the silo with a sequence of regular polygons is the position of the polygons in relation to the sides and corner of the barn. Although many choices are possible, **figures 3** and **4** illustrate two orientations that readily lend themselves to the calculation of the area. The sequences of areas that result from these two orientations are different, but each sequence approaches the same limit, the grazing area around the circular silo.

Students must also consider whether to fix the radius of the polygons at 10 feet and let the perimeter approach 20π feet as the number of sides increases or to fix the perimeter (i.e., the length of the chain) at 20π feet and let the radii of the polygons approach 10 feet as the number of sides increases. The first approach, holding the radius constant, uses techniques generally available to geometry and algebra students. The second approach is accessible to students at the precalculus level. In both solutions, we have used the polygon orientation shown in **figure 3**, with one side of the silo adjacent to the front of the barn.

A GEOMETRIC-ALGEBRAIC APPROACH

When the number of sides of the fixed-radius polygon is small, students may apply the special

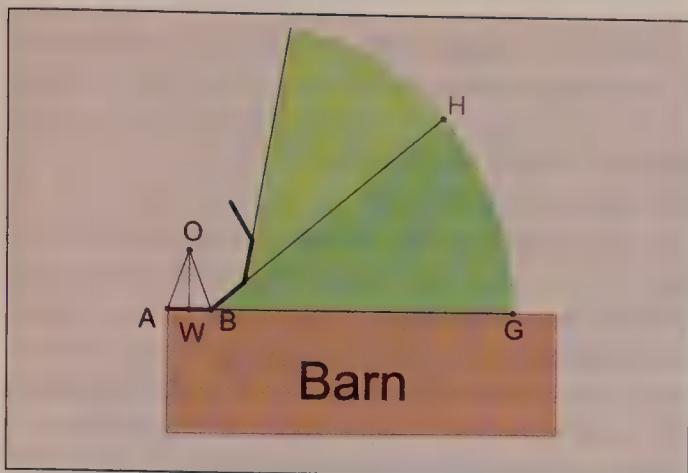


Fig. 5 Use trigonometry to find AB in the case of the regular n -gon silo with radius = 10 feet.

properties of equilateral triangles, squares, and regular hexagons. Consider the equilateral triangle in **figure 3**. Triangle AOW is a 30 - 60 - 90° triangle with $AO = 10$, implying that $AB = 10\sqrt{3}$ and $BG = 20\sqrt{3}$. (Notice that the chain length is $30\sqrt{3}$, which is shorter than the desired length of chain.) Since $m\angle GBH = 120^\circ$, the area of sector GBH is $(120^\circ/360^\circ) \cdot \pi (20\sqrt{3})^2$, or $400\pi \text{ ft.}^2$. Likewise, the area of sector HCA is $(120^\circ/360^\circ) \cdot \pi (10\sqrt{3})^2$, or $100\pi \text{ ft.}^2$. If we add these two sector areas, the goat would have $S_3 = 500\pi \text{ ft.}^2$ to graze. (The subscript 3 is associated with a triangular silo.) A square or a regular hexagon used to represent the base of the silo would lead to similar calculations.

But when students consider the pentagon and other regular polygons, the task of finding the length of a side of the polygonal silo requires them to use techniques of right-triangle trigonometry. The geometric-algebraic solution uses trigonometry to develop a generalized solution for a regular n -gon and finds the limit of the sequence of the grazing areas using a calculator.

Figure 5 shows several sides of a regular n -gon inscribed in a circle of radius 10 and suggests the sector of grass associated with each side. Each sector of grass is bounded by a circular arc that subtends an exterior angle of the regular n -gon. Because exterior angles of a regular n -gon have the same measure as the central angle (see $\angle AOB$ in **fig. 5**), then each exterior angle has measure $360^\circ/n$.

To find the length of the side AB of the n -gon in **figure 5** and, subsequently, its perimeter, construct altitude OW and note that $m\angle AOW = (1/2)m\angle AOB = 180^\circ/n$. Therefore, we get $\sin\angle AOW = AW/AO$, or, equivalently, $\sin(180^\circ/n) = AW/10$. Therefore, $AW = 10\sin(180^\circ/n)$, implying that $AB = 20\sin(180^\circ/n)$. The length of the chain, AG , is the perimeter of the n -sided polygon—that is, $n \cdot 20\sin(180^\circ/n)$.

Now, note that the radius of the first sector GBH is $AG - AB$, or $(n - 1) \cdot 20\sin(180^\circ/n)$, and that each successive radius will be decreased

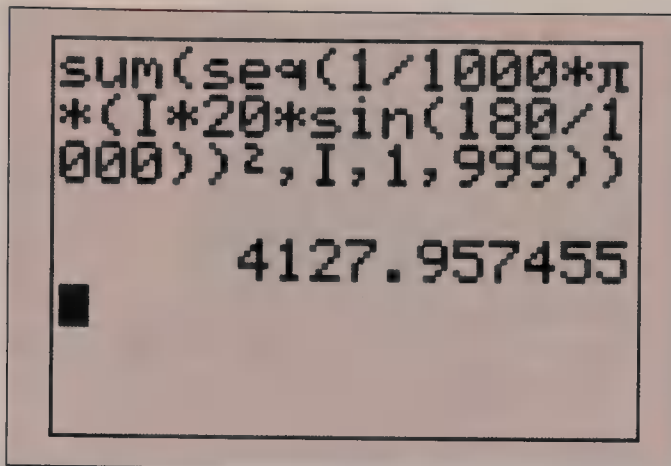


Fig. 6 For $n = 1000$, the area estimate is 4128 ft.^2 (as shown on the TI-84 Plus), which is very close to the actual area.

by another AB . Thus, the radii of the $n - 1$ sectors are $(n - 1) \cdot 20\sin(180^\circ/n)$, $(n - 2) \cdot 20\sin(180^\circ/n)$, $\dots$, $(2) \cdot 20\sin(180^\circ/n)$, $(1) \cdot 20\sin(180^\circ/n)$. The central angle subtended by each sector has measure $\theta = 360^\circ/n$, so the area of a sector with radius R is

$$\frac{360^\circ/n}{360^\circ} \cdot \pi R^2 = \frac{\pi R^2}{n}.$$

Therefore, summing the areas of the $n - 1$ sectors of grass, we get

$$S_n = \sum_{i=1}^{n-1} \frac{\pi}{n} \left(i \cdot 20\sin \frac{180^\circ}{n} \right)^2.$$

Once students see the pattern for the sum of the sectors, they can use a calculator to evaluate the sum for particular large values of n to achieve an approximation of the area when the silo is circular. For instance, on a TI-84 students can add the 99 sectors of grass when $n = 100$, by entering `sum(seq(1/100*pi*(X*20*sin(180/100))^2,X,1,99))` to obtain an answer of around 4071 ft.^2 . By simply editing this command, they can quickly find the approximation when $n = 1000$ (see **fig. 6**).

For $n > 1000$, the command used above does not work because the TI-84 Plus limits the size of lists to 999 terms. To get around this limitation, students might want to learn how to write a program on the calculator, another useful outcome of engaging in the Grazing Goat project.

A PRECALCULUS APPROACH

In our final solution, we will again orient the polygonal silo so that one of its sides (AB) is adjacent to the wall of the barn. This time, we fix the perimeter of the n -sided polygon to be 20π feet, equal to the length of the chain AG . Thus, each side of the polygon measures $20\pi/n$ ft. As the chain wraps around the silo, $n - 1$ sectors are created whose total area equals the area of the field.

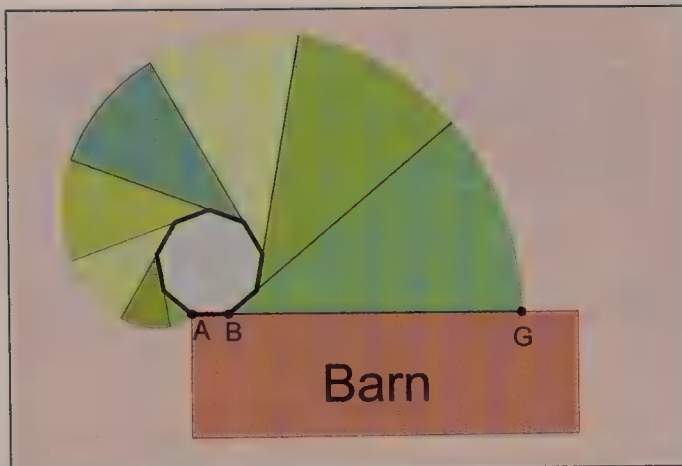


Fig. 7 An n -gon silo adjacent to the barn wall produces $n-1$ sectors with radii shrinking at each stage by the length of the side of the polygon.

Figure 7 shows the eight sectors that comprise the field when $n = 9$. If we let r_i equal the radius of the i th sector, then $r_1 = (n - 1) \cdot 20\pi/n$, $r_2 = (n - 2) \cdot 20\pi/n$, $\dots$, $r_{n-1} = 20\pi/n$. The i th sector has an area $\pi r_i^2/n$.

Summing the areas of the $(n - 1)$ sectors and substituting for r_i gives

$$\begin{aligned} S_n &= \sum_{i=1}^{n-1} \frac{\pi (r_i)^2}{n} \\ &= \sum_{i=1}^{n-1} \frac{\pi \left(i \cdot \frac{20\pi}{n} \right)^2}{n} \\ &= \frac{20^2 \pi^3}{n^3} \sum_{i=1}^{n-1} i^2 \\ &= \frac{400\pi^3}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6} \end{aligned}$$

We have used a formula for the sum of squares given in most precalculus textbooks. As the number of sides increases, the polygons of perimeter 20π feet converge to a circle of radius 10 feet, exactly the dimensions of the silo given in the problem.

Whereas geometry and algebra students might approximate a solution by using a calculator to compute S_n for large values of n , precalculus students may use methods for finding the limit of S_n as n approaches infinity. One method is illustrated in the following argument. We first expand to see that

$$\begin{aligned} \frac{(n-1)(n)(2n-1)}{6n^3} &= \frac{2n^3 - 3n^2 + n}{6n^3} \\ &= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \end{aligned}$$

As n gets large, this expression approaches $1/3$. Thus, as n approaches infinity, S_n converges to

$400\pi^3/3 \text{ ft.}^2$, which is exactly the area found in our first calculus-based solution.

ORCHESTRATING PROJECTS IN THE CLASSROOM

As a math coach and in my work at the state level, I encourage teachers of grades 5–12 to offer project-based problem-solving and technical writing experiences. These rich experiences give students an opportunity to develop the Mathematical Practices of the Common Core State Standards, which are very similar to the NCTM mathematics processes of the 1990s. It is thrilling to see students become engaged in their explorations, and I have been quite impressed with the solutions they achieve.

Teachers often raise the question of scaffolding. How many hints should be given to the students, and how should the classroom exploration be organized? Real problem solving involves frustration—temporary, we hope—as students explore different routes to a solution. Students need to develop perseverance in problem solving. And yet there may be times when students do not see a way to attack a problem. Depending on the class's progress in solving the problem, I may suggest that students begin by investigating the perimeters and areas of regular polygons as the number of sides increases.

My approach is to make the project a six- to nine-week activity. I may allow students to work periodically during class, but most work must be done outside class. During the first week, I do not allow students to work together at all; I want each person to experience problem solving individually. At the end the first week, each person must turn in a progress report with a log of the ideas that he or she has tried. If students still have not been successful, they may begin working in groups. Teachers concerned with fairness may worry that a group that includes the “star” problem solver has an advantage over other groups, so I allow the groups to interact with one another. Periodically, I expect each student to update his or her progress report. I may interview groups if their strategy is unclear to me. These interviews also provide an opportunity for me to motivate stragglers.

The goal of the project should not be limited to problem solving. Students need to develop their technical writing skills. Although students may work in groups, I expect each one to write his or her own paper. We review examples of technical writing, and the class discusses geometric diagrams, calculations, and explanations that are most appropriate. I allow students to rewrite their papers and will offer written or verbal comments to help them improve their presentation. I understand that students need training in technical writing. As usual,

time is the enemy, and projects take time. But as students gain more practice, they become more proficient in their problem-solving and technical writing skills.

After students find one solution, I encourage them to look for a second and a third solution. I encourage students to compare their work. For example, once students can find the approximate area using the **figure 3** orientation, they might try using the **figure 4** orientation. Or instead of holding the radii of the polygons constant at 10 feet, they could fix the length of the chain at 20π feet. Students begin to recognize that there are many approaches to the same problem. If the teachers within a school continue to use a particular problem at each successive grade level, students will be able to see the connections within the various branches of mathematics.

Students engaged in projects and rich problems

of this nature will benefit in many ways. They will see a range of ideas for representing problems and interact with peers yet produce their own written explanations. Add to this a yearly progression into higher mathematics, and students will become aware of the power of mathematics and enlarge their mathematical field and zone of proximal interest. Such opportunities will enrich mathematics education and support students by providing them with a more holistic experience of doing mathematics.



RICK HAVENS, rickhavens@embarqmail.com, taught high school mathematics for the Jefferson Area Schools in Jefferson, Ohio, for forty years. He is a math coach for the Formative Assessment for the Middle School (FAMS) project sponsored by the Ohio Department of Education.

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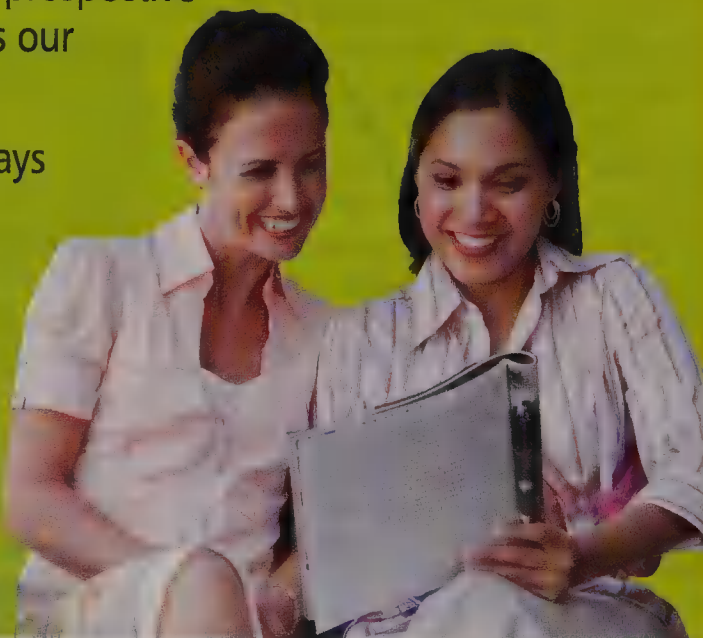
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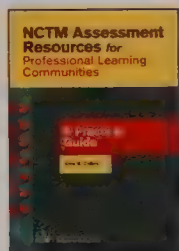


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NCTM Assessment Resources for Professional Learning Communities: A Practical Guide, Anne M. Collins, 2012. 55 pp., \$20.95 paper. ISBN 978-0-87353-685-1. Stock no. 14093. National Council of Teachers of Mathematics; www.nctm.org.



This resource guide is designed to provide structure for assessment-focused meetings of professional learning communities and give teacher leaders, mathematics

supervisors, curriculum coordinators, teachers, and professional developers a comprehensive guide for using the assessment resources that NCTM has published in recent years. After a brief overview of the foundational components of an effective comprehensive assessment system, the author suggests multiple formats of workshops for teachers' professional development regarding these components.

Two-hour workshops, which may be sequenced in alternative ways, depending on desired learning outcomes, are designed to engage teachers in a comprehensive examination of assessment, both summative and formative. Workshops are titled by topic, and workshop

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

descriptions incorporate best practices for teachers' professional development. For each workshop, the author fully describes resources (primarily, recent research readings on assessment) and materials needed, suggests activities and group work, provides possible discussion topics (including evidence of students' learning), and presents many rich mathematics tasks and how to use them); also included are notes to facilitators and assignments.

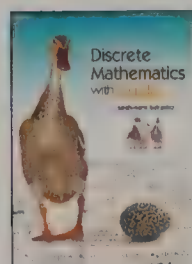
This resource guide carefully examines how the wealth of knowledge and publications from experts in formative and summative assessment might be used to support teachers in professional learning communities. Teachers who are working independently will also benefit from following the chapters' guidelines to improve their understanding of assessment.

—LuAnn Malik

Chapel Hill-Carrboro City Schools
Chapel Hill, NC

FROM OTHER PUBLISHERS

Discrete Mathematics with Ducks, Sarah-Marie Belcastro, 2012. 580 pp., \$59.95 cloth. ISBN 978-1-4665-0499-8. CRC Press; www.crcpress.com.



Discrete Mathematics with Ducks provides “serious mathematics treated with levity,” in accordance with the author’s belief that students might learn

mathematics better when it is presented “in a goofy way.” Display quotations throughout include these: “Hey! You! Don’t read any further unless you have worked through the problems in Section . . . I mean it!” and “Negation plays nicely with other connectives. . . .” In an application of logical thinking, the author discusses truth-telling ducks and

lying ducks and then quips, “That’s the unvarnished truth of the situation. But, of course, a lying duck lies (duh).”

The book is intended to be a survey course, with the intent of exposing students to a variety of concepts. Weaker mathematics students can find success using this book, the author notes, but challenging problems are included for stronger students. The approach taken is one of discovery followed by explanation of relevant material.

The book’s strengths include some good explanations, examples of good and bad ways of accomplishing tasks, and instructor notes. An algorithm showing Russian-style multiplication did not work (I tried it).

The blatant error found in the example mentioned and some of the unconventional language used make me hesitant to recommend this book.

—Marsha Guntharp

Palm Beach Atlantic University
West Palm Beach, FL

In the Dark on the Sunny Side: A Memoir of an Out-of-Sight Mathematician, Larry W. Baggett, 2012. 206 pp., \$50.00 cloth. ISBN 978-0-88385-581-2. Mathematical Association of America; www.maa.org.



As teachers, we may never fully realize how the little things that we do influence our students’ understanding of mathematics. As I read Baggett’s memoir of

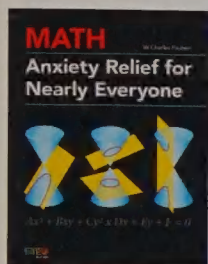
becoming a successful mathematician despite losing his sight in a childhood accident, I was reminded that the little things that we do to help our students learn can have a huge impact on how and what they understand about the world of mathematics, including their confidence in their mathematical abilities.

Reading Baggett’s memoir also helped me see life and learning mathematics as well as other subjects through the eyes

of a student who had many learning challenges to overcome but succeeded despite the hurdles.

—Kristine Butz
Cocalico High School
Denver, PA

Math Anxiety Relief for Nearly Everyone, W. Charles Paulsen, 2012. 276 pp., \$39.95 paper. ISBN 978-0-615-50540-4. STEM Books; www.STEMbooks.com.



Math Anxiety Relief for Nearly Everyone will not be viewed as a relief by mathematics teachers. Written by a mechanical engineer, the book intends to

“make math simple” by ensuring that the “casual use of math is sufficient” by avoiding “unnecessarily confusing symbolism.”

The mathematics included spans two years of algebra, trigonometry, matrix algebra, precalculus, and calculus through integration and differential equations—all in fewer than 250 pages! Throughout this broad excursion, the author’s focus is on establishing rules, with minimal attention to conceptual understanding or even the use of the rules within problem-solving contexts.

Along the way, the book offers historical inaccuracies (e.g., Archimedes’s determination of pi’s value by exhaustion) and makes mathematical errors (e.g., it states that any number divided by 0 is infinity and that any number raised to the power of zero is 1). The author perhaps would regard these as symptomatic of “unnecessarily confusing symbolism.”

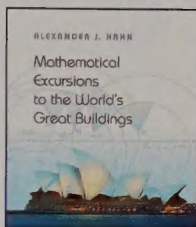
Paulsen, a mathematics teacher at a community college, is trying to help struggling students by providing clear presentations. However, by focusing on rule-based content (summarized in a collection of Excel routines in an appendix), he falls short of promoting deeper-level understanding and problem solving—that is, teaching mathematics.

This book should be avoided by both mathematics teachers and students. Already too many students focus on mastering rules and then experience “anxiety” in future mathematics classes,

a result of their lack of understanding amid assumed competence. The fault lies not in the mathematics but in the students’ assumption that they were in fact learning mathematics.

—Jerry Johnson
Western Washington University
Bellingham, WA

Mathematical Excursions to the World’s Great Buildings, Alexander J. Hahn, 2012. 352 pp., \$49.50 cloth. ISBN 978-0-691-14520-4. Princeton University Press; www.press.princeton.edu.



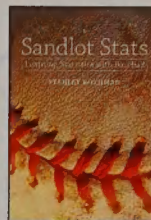
The engaging theme of architecture is woven through this book, which is suitable for standard or advanced mathematics history or survey courses. Hahn’s use of physics and engineering concepts presents readers with numerous examples of how important mathematics is in other areas outside the classroom. The problem sets are challenging and require readers to use analytical thinking. This book would also serve well as a supplemental text for introducing analytical and quantitative reasoning skills as well as providing numerous examples of fun challenge questions and mathematics and science trivia.

I recommend this book for advanced survey courses or special topics courses. I would not recommend it for a lower-level survey course for nonmathematics majors because the mathematics and concepts involved are too advanced and too specialized into one area (architecture) to serve a standard undergraduate survey course well.

—Craig McBride
University of Puget Sound
Tacoma, WA

Sandlot Stats: Learning Statistics with Baseball, Stanley Rothman, 2012. 586 pp., \$85.00 cloth. ISBN 978-1-4214-0602-2. The Johns Hopkins University Press; www.press.jhu.edu.

Sandlot Stats is a college-level introductory statistics textbook that uses baseball as its focus. The book consists of eighteen chapters and three appendices and includes a diverse selection of examples



and exercises. Rothman incorporates spreadsheet simulations and uses online baseball data sources throughout. He also explores a number of interesting and hotly debated questions in baseball: Which was the best team of all time? At what age does a player reach his peak? Is there a home-field advantage?

Baseball fans will especially enjoy Rothman’s comparison of the batting performance of Hank Aaron and Barry Bonds (chapter 4) and his analysis of hitting streaks (chapter 16). Inferential statistics topics such as confidence intervals and hypothesis testing for one and two population parameters are covered, but the overall emphasis is descriptive statistics. The book is very readable and well organized; however, a number of the graphs and figures could have benefitted from additional copyediting.

High school statistics teachers could use this book as a course supplement or an enrichment source for sports-crazy students. Any college professor teaching a sports statistics course or looking for ways to enliven a traditional course would also find it interesting. Even baseball fans wanting to learn more about descriptive measures used in the game could benefit from Rothman’s interesting exposition.

—Marc Michael
Frostburg State University
Frostburg, MD

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MY FAVORITE
lesson

Kevin D. Winiarz

Complicate *This!*

What exactly are we supposed to do when the directions say, “Simplify?”

One subtle yet nearly ubiquitous difficulty for mathematics students involves understanding directions. My response to this second-year algebra student’s question was that *simplify* means to put the term or expression in its most basic form by performing all operations and reductions. Although satisfactory for this student’s specific query, the answer to this question is neither easy nor absolute; it depends on context and aesthetic preference. Browse some mathematics dictionaries and carry out an Internet search to see for yourself.

Students’ difficulty in correctly comprehending the meaning of terse, somewhat cryptic instructions is not uncommon and may explain why some students mistakenly attempt to solve for a variable in an expression when they intend to simplify. Directions to simplify an expression such as $4 \cdot (9 - 3) + 2$ seldom lead to confusion or errors, but trouble occurs when the expression contains a variable, such as $3x - 4 + 2(8x - 4x)$. In this case, some students try, mistakenly, to solve for x .

To help students understand the distinction between solving and simplifying, I use the following tactic. On the board, I write something like this:

Simplify: $10 + 3 - 5(6/2)$.

I explain what I mean by *simplify*, and the class and I complete the example together.

Next I write this:

Complicate: 5.

I explain that I want students to reverse the simplifying process with the quantity 5. That is, they are to *unsimplify*, to go backward using a chain of arithmetic reasoning and procedures to create a more elaborate, complex way to represent 5.

Initially, many students create straightforward examples, such as $2 + 3$ or $10 - 5$. I suggest that they use the identity elements of addition and multiplication as well as square roots, thus generating representations such as $\sqrt{36} - 1$. If they have not yet done so, I then encourage students to create expressions that contain one or more variables and also terms of degree greater than one, thereby incorporating algebraic representations and using properties of exponents in their creative efforts. This prompting produces representations for 5 such as

$$\frac{30x}{6x} \text{ and } \frac{(\sqrt{5}\pi y)^2}{(\pi y)^2}.$$

To conclude our complicating tasks, I ask each student to choose any whole number and complicate it, requiring at least one variable in the process. Students work for a few minutes, and then I distribute index cards on which they write the complicated expression for their chosen number. They exchange cards with classmates, and I ask each

one to simplify the classmate’s expression. After simplifying for a few minutes on their own, students pair up with the person with whom they exchanged cards and check the accuracy of their complicating and simplifying procedures. Once students have successfully completed this activity, we can extend the idea to a polynomial with one or more variables—for example, “Complicate: $3x + 5$.”

This exercise is analogous to disassembling and assembling an object such as a bicycle or a bookshelf using a given set of instructions. If a person can take an object apart and put it back together, or vice versa, he or she likely has a reasonable understanding of the processes at work. The simplify-complicate exercise allows students to accept more readily an agreed-on meaning for the phrase *simplify*. A bidirectional context emphasizes the fundamental properties and mechanics of arithmetic and algebra.

Have your students complicate some mathematics!



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The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by **Roger Day**, day@ilstu.edu,
Illinois State University, Normal, IL

CORRECTION

“Modeling Exponential Decay” (The Back Page, *MT* December 2013/January 2014, vol. 107, no. 5, p. 400) shows significant similarity to work by Henri Picciotto. Consequently, Mr. Picciotto’s work should have been added to the reference list: <http://www.mathedpage.org/calculator/index.html#rolling>.

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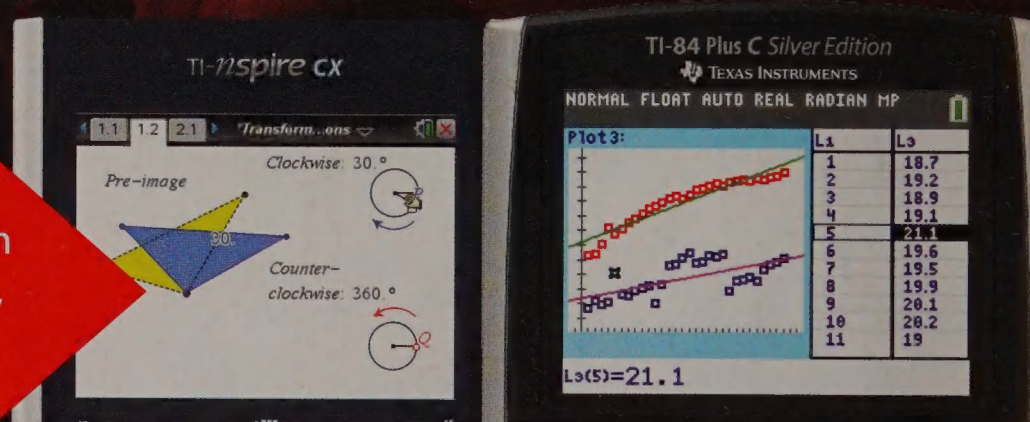
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